INFERENCES BASED ON DOUBLY CENSORED SAMPLES FROM EXPONENTIAL DISTRIBUTIONS

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By
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to the

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To

My beloved father

And

The memory of my mother

13/12/84

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CERTIFICATE

This is to certify that the matter embodied in the thesis entitled "INFERENCES BASED ON DOUBLY CENSORED SAMPLES FROM EXPONENTIAL DISTRIBUTIONS" by Mr. Narayana Shetty B. for the award of the Degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur, is a record of bonafide research work carried out by him under my supervision and guidance. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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SYNOPSIS

This study is concerned with the problem of estimating the parameters and testing the equality of location parameters θ_i ($i = 1, 2, \ldots, K$) of $K (\geq 2)$ two-parameter exponential distributions $E(\theta_i, \sigma)$ based on K independent type II censored samples. Here the common scale parameter σ is assumed to be unknown. Type II censored sample is an ordered sample in which a known number of smallest (left) observations and/or largest (right) observations are missing. An ordered sample is obtained by rearranging the variates in an ascending order of magnitude.

Most of the work in this field is done when complete or right censored samples are available. However, there are situations when some smallest observations are also not available. In the present work, main attention has been paid to type II doubly censored samples.

Let $X^{(i)}_{r_i+1}, X^{(i)}_{r_i+2}, \ldots, X^{(i)}_{n_i-s_i}$ ($i = 1, 2, \ldots, K$) be K independent ordered samples from $E(\theta_i, \sigma)$ with $r_i \geq 0$, $s_i \geq 0$ and $r_i+1 \leq n_i-s_i$. Based on these observations, the Least Square (LS) and Maximum Likelihood (ML) estimators of θ_i ($i = 1, 2, \ldots, K$) and σ are obtained. For $K = 2$, the LS and ML estimators are derived under the assumption $\theta_1 = \theta_2 = \theta$. Some distribution theory results regarding these estimators are also obtained. A brief comparison of the LS and ML estimators is made by using the mean square error criterion.

For $K = 2$, the test statistics based on LS and ML estimators are proposed for testing the null hypothesis $\theta_1 = \theta_2$ against one-sided alternatives. These statistics are equivalent to

$$T = \{X^{(1)}_{r_1+1} - X^{(2)}_{r_2+1}\}/\sigma^* ,$$

where

$\sigma^* = P_1/d$ is the pooled estimator of σ,

$$P_1 = \sum_{i=1}^{K} \{ \sum_{j=r_i+1}^{n_i-s_i} X_j^{(i)} + s_i X^{(i)}_{n_i-s_i} - (n_i-r_i) X^{(i)}_{r_i+1}\},$$

$$d = \sum_{i=1}^{K} (n_i-r_i-s_i-1) \text{ and } K = 2.$$

Against the alternative $\theta_1 > \theta_2$, large values of T lead to the rejection of the null hypothesis. The null and non-null distributions of T are derived. Some exact and approximate critical points of T are tabulated. Power values along with a normal approximation are also tabulated. From this, it is concluded that the test T is more sensitive for the left censoring than for the right censoring.

The test statistics V_1, V_2 and λ based on LS estimators, ML estimators and LR test procedures respectively, are proposed for testing $\theta_1 = \theta_2$ against $\theta_1 \neq \theta_2$. These are given by

$$V_i = |T-q_i| \qquad (i = 1,2),$$

and for $Y \geq 0$,

$$\lambda = R\{\frac{\hat{\sigma}}{\hat{\sigma}_o}\}^{d*} \frac{Y^{r_1+r_2} (d^* \hat{\sigma}_o + fY - P)^f \exp(-P/\hat{\sigma}_o)}{\{d^* \hat{\sigma}_o + (r_1+f)Y - P\}^{r_1+f} \{P - d^* \hat{\sigma}_o + r_2 Y\}^{r_2}},$$

where $q_1 = b_1 - b_2$, $q_2 = m_1 - m_2$,

$$b_i = \sum_{j=1}^{r_i+1} (n_i - j + 1)^{-1} \quad (i = 1, 2),$$

$$m_i = \log \{n_i/(n_i - r_i)\} \quad (i = 1, 2),$$

$$\hat{\sigma} = P_1/d^*, \quad d^* = d+2,$$

$$f = n_1 + n_2 - r_1 - r_2,$$

$$R = \{\prod_{i=1}^{2} n_i^{n_i} (n_i - r_i)^{-(n_i - r_i)}\} \exp(d^*), \quad Y = X_{r_2+1}^{(2)} - X_{r_1+1}^{(1)}$$

$$P = \sum_{i=1}^{2} \{ \sum_{j=r_i+1}^{n_i - s_i} X_j^{(i)} + s_i X_{n_i - s_i}^{(i)} - (n_i - r_i) X_{r_1+1}^{(1)} \}$$

and $\hat{\sigma}_o$ is the solution of the equation

$$e^{Y/\hat{\sigma}_o} = \{1 + \frac{r_2 Y}{P - d^* \hat{\sigma}_o}\}/\{1 + \frac{r_1 Y}{d^* \hat{\sigma}_o + fY - P}\}.$$

The expression for λ given above is for $r_1 > 0$, $r_2 > 0$. Considerably simpler expressions are obtained if $r_1 = 0$ and/or $r_2 = 0$. For $Y < 0$, λ is obtained by replacing n_1, n_2, r_1, r_2 by n_2, n_1, r_2, r_1 respectively in the above expression.

The exact critical points and power values of the tests V_1 and V_2 are evaluated. Due to the complex nature of λ, only simulated critical points and power values of λ are tabulated. On the basis of power calculations, it is concluded that the test V_2 is somewhat more biased than V_1 and there is very little difference between V_1 and λ. Since the statistic λ is far mor complicated than V_1, use of V_1 is recommended.

For testing $K(\geq 3)$ populations, two test statistics are proposed. We first consider the case of right censoring, so tha the smallest observation of each sample is available. For simplicity of notations let $X_i = X_1^{(i)}$ $(i = 1,2,\ldots,K)$ and $X_{(1)} = \min(X_1,X_2,\ldots,X_K)$. The statistic given by

$$T_1 = \{X_1-\min(X_2,X_3,\ldots,X_K)\}/\sigma^*$$

is proposed for testing $H_0 : \theta_1 = \theta_2 = \ldots = \theta_K = \theta$ against $H_1 : \theta_1 > \max(\theta_2,\theta_3,\ldots,\theta_K)$, and the statistic

$$T_2 = \{\max_{1\leq i\leq K}(X_i) - \min_{1\leq i\leq K}(X_i)\}/\sigma^*$$

is proposed for testing H_0 against H_2 : at least one θ_i is different from θ. The exact critical points and power values of the tests T_1 and T_2 are evaluated. For $K = 3$ case, the performance of statistic T_1 is studied for different combinatio of n_1,n_2,n_3 and d. It is observed that, the test T_1 is more sensitive for changes in n_1 than changes in n_2,n_3 and d. For $K = 3$ and equal sample size case, the performance of T_1 and T_2

along with that of U_3 and U_4 is studied. The statistics U_3 and U_4 are given by

$$U_3 = [\max_{2\leq i\leq K} \{n_i(X_i-X_1),\ n_1(X_1-X_i)\}]/d\sigma^*$$

and

$$U_4 = \sum_{i=1}^{K} n_i(X_i-X_{(1)})/\{(K-1)\sigma^*\}.$$

The test U_4 is actually the LR test and U_3 has been proposed by other authors. On the basis of power calculations carried out, T_1 is recommended for testing H_o against H_1, and the LR test statistic U_4 is recommended for testing H_o against H_2.

If the smallest observation is missing in each sample of size n, then the following test statistics are studied :

$$V_1 = \{X_2^{(1)} - \min_{2\leq i\leq K} (X_2^{(i)})\}/\sigma^*$$

for testing H_o against H_1;

$$V_2 = \{\max_{1\leq i\leq K} (X_2^{(i)}) - \min_{1\leq i\leq K} (X_2^{(i)})\}/\sigma^*,$$

and

$$V_3 = [\max_{2\leq i\leq K} \{(X_2^{(i)}-X_2^{(1)}),\ (X_2^{(1)}-X_2^{(i)})\}]/\sigma^*$$

for testing H_o against H_2. For K = 3, exact critical points and simulated power values of these tests are tabulated. On the basis of these calculations V_1 is recommended for testing H_o against H_1, but nothing can be said regarding the preference of V_2 over V_3, as in some cases V_2 performs better and in some cases V_3 performs better.

CHAPTER I

INTRODUCTION AND SUMMARY

1.1 Scope.

Exponential distribution is often proposed for modelling the lifetime distributions of items like electronic components, mechanical breakdowns, light bulbs etc. [Davis (1952); Epstein (1958); Proschan(1963); Nelson (1975)]. In a two-parameter exponential distribution, the location parameter is interpreted as the minimum (or guarantee) time, before which no failures occur, and the scale parameter, as the mean life measured from the location parameter as the starting point.

There are several situations, where the complete sample is neither available nor desirable. Since life-testing experiments are usually destructive, this limits the number of items to be tested (Sinha and Kale 1980, p. 18). Moreover, in the ordered samples frequently found in biological data either some smallest and/or some largest observations are not available [Ipsen (1949)]. An ordered sample is obtained by rearranging the variates in an ascending order of magnitude. In an ordered sample, if a known number of smallest (left) values or largest (right) values or both are missing, then such a sample constitutes a type II censored sample.

Several problems dealing with the estimation and testing of hypothesis of a two-parameter exponential distribution are

discussed in several books, for example see Sarhan and Greenberg (1962), Mann, Schafer and Singpurwalla (1974), Bain (1978), Sinha and Kale (1980). Complete and right censored samples have been considered by many authors [Walsh (1950), Halperin (1952), Epstein and Sobel (1953), Hogg and Tanis (1963), Grubbs (1971), Kumar and Patel (1971), Dubey (1973), Weinman et al. (1973), Khatri (1974), Perng (1978), Mathai (1979), Regal (1980), Bhattacharyya and Mehrotra (1981), Hsieh (1981), Goria (1982), Mehrotra and Bhattacharya (1982), Singh (1983), Singh and Narayan (1983) etc.].

Although the left censored samples have not been considered so thoroughly as the right censored samples, there are situations in which some smallest observations are not available. For example, in experimental biology, n animals are tested for antibodies after a certain period of time. Only (n-r) of these samples contain measurable amounts while r of the animals develop the antigen at a level too low for measurement by the prevailing technique (Ipsen, 1949). This gives rise to a censored sample from left. Another example where the smallest order statistic is difficult to observe is the failure time of human kidney. It is not easy to tell the failure time of one kidney since noticeable symptoms occur only when both kidneys fail. Similarly, in a transistor set four battery cells may be used. Failure of only one cell may not affect the performance, and hence the first failure time may go unreported. But when two cells fail, the transistor may not

work properly and the second smallest order statistic becomes the first available observation.

Based on doubly censored samples, the Least Square (LS) estimators and Maximum Likelihood (ML) estimators of the parameters of a two-parameter exponential distribution were derived by Sarhan (1955) and Kambo (1978) respectively. Recently, Tiku (1981) and Khatri (1981) have considered the problem of testing equality of location parameters of two-parameter exponential distributions based on type II doubly censored samples.

One of the problems arising in life-testing experiments is the estimation and the comparison of minimum (or guarantee) time of the items. Related to this problem, the present work is concerned with the problem of estimating the parameters and testing the equality of location parameters of $K(\geq 2)$ two-parameter exponential distributions. Here the scale parameters are assumed to be equal but unknown. In this thesis, main attention has been paid to type II doubly censored samples.

The following topics are studied in this thesis :

1. Comparative study of the ML and LS estimators of the parameters.
2. Testing equality of location parameters against one-sided alternatives.
3. Testing equality of location parameters against two-sided alternative.

4. Generalized statistics for K right censored samples.
5. Generalized statistics in equal sample case, when one observation is missing on the left.

The results obtained are compared with the existing results. Suitable graphs and tables are provided to support the theory, wherever necessary. The notations which are used consistently in the text are given in the next section and the subsequent sections describe briefly the above mentioned topics.

1.2. Notations and abbreviations.

As far as possible random variables will be designated by upper case letters, and their realizations (observations) by the corresponding lower case letters.

$X_1^{(i)} \leq X_2^{(i)} \leq \cdots \leq X_{n_i}^{(i)}$	ith ordered sample of size n_i $(i = 1,2,\ldots,K)$ with superscript i dropped if there is only one sample.
$\underset{\sim}{X}(i) = (X_{r_i+1}^{(i)}, X_{r_i+2}^{(i)}, \ldots, X_{n_i-s_i}^{(i)})$	ith type II censored sample in which first r_i and last s_i observations are missing, where $r_i \geq 0$, $s_i \geq 0$, and $r_i+1 \leq n_i-s_i$.
$F_X(x), P(x) = P[X \leq x]$	cumulative distribution function of X
$f_X(x)$, $p(x)$	probability density function of X
$E(X)$	mean of X
$Var(X)$	variance of X
$MSE(T)$	mean square error of T
$SE(T)$	standard error of T
LR	likelihood ratio

KP test	Kumar and Patel (1971) test
LS	least square
ML	maximum likelihood
i.i.d.	independent identically distributed
pdf	probability density function
jpdf	joint pdf
cdf	cumulative **distribution** function
w.r.to	with respect to
MVU	minimum variance unbiased
DF	degrees of freedom
$E(\theta, \sigma)$	two-parameter exponential distribution with location parameter θ and scale parameter σ
$N(\mu, \eta^2)$	normal distribution with mean μ and variance η^2
χ_ν^2	central chi-square distribution with ν DF
σ^*, θ^*	LS estimators of σ and θ respectively
$\sigma_o^*, \theta_o^*, \hat{\sigma}_o, \hat{\theta}_o$	LS and ML estimators under the hypothesis $\theta_1 = \theta_2 = \theta$.
$\underset{=}{d}$	is distributed according to
$\simeq$	is approximately equal to
$\equiv$	is identically equal to

$$B(a,b) = \int_o^1 t^{a-1}(1-t)^{b-1}\, dt, a > 0, \ b > 0$$

$$\Gamma(a) = \int_o^\infty t^{a-1} e^{-t}\, dt, \ a > 0$$

$$Q_p(x|s) = \int_x^\infty t^{p-1}e^{-t-st}\,dt/(p-1)! \ , \ x > 0, \ p = 1,2,\ldots \text{ for } s > -1$$

$$= \sum_{j=0}^{p-1} \exp\{-x(1+s)\}\{x(1+s)\}^j/\{j!(1+s)^p\}$$

$$L_p(x|s) = \int_0^x t^{p-1}e^{-t+st}\,dt/(p-1)! \ , \ x > 0, \ p = 1,2,\ldots$$

$$= \begin{cases} x^p/p! \ , & s = 1 \\ 1/(1-s)^p - \sum_{j=0}^{p-1} \exp\{-x(1-s)\}x^j(1-s)^{j-p}/j! \ , & s \neq 1 \end{cases}$$

$$d^* = \sum_{i=1}^{K} (n_i-r_i-s_i)$$

$$d = \sum_{i=1}^{K} (n_i-r_i-s_i-1) = d^* - K$$

$$P_1 = \sum_{i=1}^{K} \{ \sum_{j=r_i+1}^{n_i-s_i} X_j^{(i)} + s_i X_{n_i-s_i}^{(i)} - (n_i-r_i) X_{r_i+1}^{(i)} \}$$

$$P = \sum_{i=1}^{2} \{ \sum_{j=r_i+1}^{n_i-s_i} X_j^{(i)} + s_i X_{n_i-s_i}^{(i)} - (n_i-r_i) X_{r_1+1}^{(1)} \}$$

1.3. Comparative study of the ML and LS estimators of the parameters.

Let $X_{r+1}, X_{r+2}, \ldots, X_{n-s}$ be a type II censored sample from an exponential distribution $E(\theta,\sigma)$ with pdf

$$f(x;\theta,\sigma) = \sigma^{-1} \exp\{-(x-\theta)/\sigma\}, \ 0 < \sigma < \infty, \ \theta \leq x < \infty.$$

Sukhatme (1936) obtained the best unbiased estimators of θ and σ based on a complete sample ($r = 0$, $s = 0$ case) of size n. Lloyd (1952) discussed the technique of estimating θ and σ by applying general LS theory to an ordered sample. By applying

Lloyd's method Sarhan (1954,1955) and Greenberg and Sarhan (1962) have obtained the best linear unbiased estimators of mean and standard deviation for double and middle censoring case as well.

Let $X^{(i)}_{r_i+1}, X^{(i)}_{r_i+2}, \ldots, X^{(i)}_{n_i-s_i}$ $(i = 1,2,\ldots,K)$ be K independent type II doubly censored samples from $E(\theta_i,\sigma)$. The LS estimators θ^*_i and σ^* of $\theta_i (i = 1,2,\ldots,K)$ and σ are obtained in Section 2.2. These are given by

$$\theta^*_i = X^{(i)}_{r_i+1} - b_i\sigma^* \quad (i = 1,2,\ldots,K) \text{ and } \sigma^* = P_1/d,$$

where $b_i = \sum_{j=1}^{r_i+1} (n_i-j+1)^{-1}$ $(i = 1,2,\ldots,K)$, $d = \sum_{i=1}^{K} (n_i-r_i-s_i-1)$

and $P_1 = \sum_{i=1}^{K} \{ \sum_{j=r_i+1}^{n_i-s_i} X^{(i)}_j + s_i X^{(i)}_{n_i-s_i} - (n_i-r_i)X^{(i)}_{r_i+1}\}$.

In Section 2.3, it is shown that $X^{(i)}_{r_i+1}$ $(i = 1,2,\ldots,K)$ and σ^* are independent, and $2d\sigma^*/\sigma$ has a χ^2_{2d} distribution.

ML estimators of the parameters for a complete and right censored sample were derived by Sukhatme (1936) and Epstein and Sobel (1954) respectively. For a doubly censored sample, ML estimators were discussed by Tiku (1967). He has obtained the ML equations and has mentioned that, those equations do not have explicit solutions. Hence, he obtained a modified ML estimator of the parameters. Using these equations, Kambo (1978) has obtained explicit expressions for the ML estimators. In Section 2.4, the ML estimators $\hat{\theta}_i$ and $\hat{\sigma}$ of θ_i $(i = 1,2,\ldots,K)$ and σ are obtained. These are given by

$\hat{\theta}_i = X^{(i)}_{r_i+1} + \hat{\sigma} \log (1-r_i/n_i)$ $(i = 1,2,\ldots,K)$ and $\hat{\sigma} = P_1/d^*$, where $d^* = d+K$.

Kambo (1978) compared the minimum variance unbiased (MVU) estimators with the ML estimators, when a doubly censored sample is available. He has shown that for a single sample $MSE(\sigma^*) > MSE(\hat{\sigma})$ and verified numerically that $MSE(\theta^*)$ can be greater or less than $MSE(\hat{\theta})$. By this, he concluded that, some times ML estimators are better than MVU estimators. Similar type of comparison is done in Section 2.5 and it is concluded that in general for $K \geq 3$, $MSE(\sigma^*)$ is less than $MSE(\hat{\sigma})$. However, for $K \leq 2$, $MSE(\sigma^*)$ is greater than $MSE(\hat{\sigma})$.

Epstein and Tsao (1953) derived the ML estimators under the hypothesis $\theta_1 = \theta_2 = \theta$ for right censored samples. In Section 2.6, the LS estimators and ML estimators for doubly censored samples are discussed under the same hypothesis. A brief comparison has been done between LS estimators and ML estimators in this case also.

1.4. Testing equality of location parameters against one-sided alternatives.

For the right censored samples from two populations Kumar and Patel (1971) have proposed a test statistic for testing $H_o : \theta_1 = \theta_2$ against $H_2 : \theta_1 \neq \theta_2$. Weinman *et al.* (1973) extended it for testing H_o against one-sided alternative $H_1' : \theta_1 < \theta_2$. Their statistic is $W = (X_1^{(2)}-X_1^{(1)})/\sigma^*$, where

$\sigma^* = P_1/d$ is the pooled estimator of σ. They obtained the critical point c as

$$c = \begin{cases} d[\{n_1/(n_1+n_2)\alpha\}^{1/d}-1]/n_2 & \text{if } n_1 \le \alpha n_2/(1-\alpha) \\ d\,[1-\{n_2/(n_1+n_2)(1-\alpha)\}^{1/d}]/n_1 & \text{otherwise,} \end{cases}$$

where α is the chosen level of significance. Note that $c \geq 0$ for $n_1 \geq \alpha n_2/(1-\alpha)$. They obtained the power function $P(\varphi)$ of W for $c \geq 0$ and $\varphi = (\theta_2-\theta_1)/\sigma \geq 0$ as

$$P(\varphi) = 1-\{e^{-d\varphi/c}/(n_1+n_2)\}\,[(n_1+n_2)\sum_{i=0}^{d-1}(d\varphi/c)^i/i!$$

$$-\{n_1/(1+n_2c/d)^d\}\sum_{i=0}^{d-1}\{\varphi(d+n_2c)/c\}^i/i!\,]$$

$$-\,[n_2e^{-n_1\varphi}-n_2e^{-d\varphi/c}\sum_{i=0}^{d-1}\{\varphi(d-n_1c)/c\}^i/i!\,]$$

$$\cdot\,1/\{(n_1+n_2)(1-n_1c/d)^d\},$$

provided that $n_1c \neq d$. For the case $n_1c = d$, the last term in $P(\varphi)$ becomes

$$-n_2e^{-n_1\varphi}(d\varphi/c)^d/(n_1+n_2)d!,$$

For $c < 0$, $P(\varphi)$ is given by

$$P(\varphi) = 1-n_2\,\exp(-n_1\varphi)\,(1-n_1c/d)^{-d}/(n_1+n_2).$$

In Chapter III, a test statistic based on ML and LS estimators is proposed for testing H_0 against $H_1 : \theta_1 > \theta_2$. This test is equivalent to

$$T = \{X^{(1)}_{r_1+1} - X^{(2)}_{r_2+1}\}/\sigma^*.$$

The distributions of the statistic T under H_o and H_1 are derived. Approximations for null distribution in terms of Student's t and normal distributions are studied. Approximate critical points obtained from above approximate null distributions are also tabulated along with the exact critical points. It is observed that the normal approximation is better if $r_1 > r_2$, otherwise Student's t approximation is better. Some exact and normally approximated power values are tabulated. Also, the variation in power due to different combination of r_1 and r_2 is plotted. With power function as a base, it is concluded that the test T is unbiased and it is more sensitive for the variations in r_1 compared to the variations in r_2. Further, the performance of the test is not seriously affected for variations in right truncations for fixed values of n_1, n_2, r_1 and r_2.

1.5. Testing equality of location parameters against two-sided alternative.

Epstein and Tsao (1953) discussed the LR test procedure for testing various types of hypotheses based on right censored samples. Kumar and Patel (1971) proposed a test based on $|(X_1^{(1)}-X_1^{(2)})/\sigma^*|$ for testing $H_o : \theta_1 = \theta_2$ against $H_2 : \theta_1 \neq \theta_2$. Dubey (1973) and Weinman et al. (1973) derived the power function of KP test. Weinman et al. compared the KP test with LR test and they concluded that in general LR test performs better than KP test. The power function $P(\varphi)$ of KP test with critical point c was obtained by Weinman et al. For $\varphi = (\theta_2-\theta_1)/\sigma \geq 0$

it is given by

$$P(\varphi) = 1-\exp(-d\varphi/c) \sum_{i=1}^{d-1} (d\varphi/c)^{i}/i!$$

$$+ \{n_2/(n_1+n_2)\} \exp(-n_1\varphi)\{(1+n_1 c/d)^{-d}-(1-n_1 c/d)^{-d}\}$$

$$+ \{n_1/(n_1+n_2)\}(1+n_2 c/d)^{-d} e^{-d\varphi/c} \sum_{i=0}^{d-1} \{\varphi(n_2+d/c)\}^{i}/i!$$

$$+ \{n_2/(n_1+n_2)\}(1-n_1 c/d)^{-d} e^{-d\varphi/c} \sum_{i=0}^{d-1} \{\varphi(-n_1+d/c)\}^{i}/i!$$

for $1-n_1 c/d \neq 0$. However, for $1-n_1 c/d = 0$ the third and last terms of $P(\varphi)$ become respectively

$$2^{-d}\{n_2/(n_1+n_2)\}\exp(-n_1\varphi) \text{ and } -n_2(d\varphi/c)^{d} \exp(-n_1\varphi)/\{d!\,(n_1+n_2)\}.$$

The power $P(\varphi)$ for $\varphi < 0$ is obtained by interchanging n_2 and n_1 and evaluating the above expression for $|\varphi|$.

Recently, Tiku (1981) considered the problem of testing $H_o : \theta_1 = \theta_2$ against $H_2 : \theta_1 \neq \theta_2$ based on doubly censored samples. He proposed the test statistic

$$U = |\{X^{(1)}_{r_1+1} - X^{(2)}_{r_2+1}\}/\sigma^*|$$

for testing H_o against H_2 and obtained its null distribution as

$$f_U(u) = H\Big[\sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(f+j, r_1+1)\{1+h_2(j)u\}^{-d-1}$$

$$+ \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(f+j, r_2+1)\{1+h_1(j)u\}^{-d-1}\Big], \quad u > 0,$$

where $H = \prod_{i=1}^{2}\{B(n_i-r_i, r_i+1\}^{-1}$, $h_i(j) = (n_i-r_i+j)/d \quad (i = 1, 2.),$

$$f = n_1+n_2-r_1-r_2 \text{ and } d = f-2-s_1-s_2.$$

Khatri (1981) has derived the non-null distribution of U as follows :

$$g_U(u) = H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(f+j, r_2+1) \sum_{i=0}^{d} (d\varphi/u)^i \{1+h_1(j)u\}^{i-d-1}$$
$$\cdot \exp(-d\varphi/u)/i! + H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(f+j, r_1+1) \{1+h_2(j)u\}^{-d-1}$$
$$\cdot \exp\{-h_2(j)d\varphi\} - H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(f+j, r_1+1) \{1-h_2(j)u\}^{-d-1}$$
$$\cdot \exp\{-d\varphi/u\} \left[\sum_{i=1}^{d} (d\varphi/u)^i \{1-h_2(j)u\}^i / i! \right], \; u \geq 0,$$

where $\varphi = (\theta_1-\theta_2)/\sigma \geq 0$. This final expression is slightly wrong due to some integration errors. Further, he has not taken into account the singularities at $u = 1/h_2(j)$ for $j = 0, 1, 2, \ldots, r_2$ (for $\varphi > 0$), as has been done by Weinman et al. (1973) for right censored samples. We give correct form of this expression in Section 4.2.

In Chapter IV, two statistics defined by $V_1 = |T-q_1|$ and $V_2 = |T-q_2|$ based on LS and ML estimators respectively are proposed for testing H_o against H_2, where

$$T = (X^{(1)}_{r_1+1} - X^{(2)}_{r_2+2})/\sigma^*, \; q_1 = b_1-b_2, \; q_2 = m_1-m_2,$$

$$b_i = \sum_{i=1}^{r_i+1} (n_i-j+1)^{-1} \; (i = 1, 2) \text{ and } m_i = \log\{n_i/(n_i-r_i)\} \, (i=1, 2).$$

The null and the non-null distributions of these statistics are derived. As a special case, the non-null distribution of Tiku's statistic(U) is also obtained.

The LR test statistic λ is derived for testing H_o against H_2. For $r_1 > 0$, $r_2 > 0$, $Y \geq 0$, it is given by

$$= \text{const.} \left[\frac{\hat{\sigma}}{\hat{\sigma}_o}\right]^{d^*} \frac{Y^{r_1+r_2}(d^* \hat{\sigma}_o+fY-P)^f \exp(-P/\hat{\sigma}_o)}{\{d^*\hat{\sigma}_o+(r_1+f)Y-P\}^{r_1+f}\{P-d^* \hat{\sigma}_o+r_2Y\}^{r_2}},$$

where const $= n_1^{n_1} n_2^{n_2}(n_1-r_1)^{-(n_1-r_1)}(n_2-r_2)^{-(n_2-r_2)}\exp(d^*)$, $\hat{\sigma} = P_1/d^*$,

$$P_1 = \sum_{i=1}^{2} \left[\sum_{j=r_i+1}^{n_i-s_i} X_j^{(i)}+s_iX_{n_i-s_i}^{(i)}-(n_i-r_i)X_{r_i+1}^{(i)}\right],$$

$Y = X_{r_2+1}^{(2)}-X_{r_1+1}^{(1)}$, $P = P_1+(n_2-r_2)Y$ and $\hat{\sigma}_o$ is the solution of the equation

$$e^{Y/\hat{\sigma}_o} = \left[1 + \frac{r_2Y}{P-d^* \hat{\sigma}_o}\right] \Big/ \left[1 + \frac{r_1Y}{d^* \hat{\sigma}_o + fY-P}\right].$$

For $Y < 0$, λ is obtained by replacing n_1, n_2, r_1, r_2 by n_2, n_1, r_2, r_1 respectively in the above expression.

The critical points of the tests V_1, V_2, U and λ are tabulated. The comparative performance of all the four test statistics is studied. Since the distribution of λ is not easy to derive, only simulated critical points and power values are used. Using the power function as a base, it is concluded that the test U is more biased than V_1 and V_2; V_2 is more biased than V_1, and there is very little difference in the power values of λ and V_1. Since the statistic λ is very complicated, while

the statistic V_1 is considerably simple, we strongly recommend the use of the test statistic V_1 in such situations.

1.6. Generalized statistics for K right censored samples.

Khatri (1974) derived the LR test U_1 and two test statistics U_2 and U_3, using union intersection principle for testing $H_o : \theta_1 = \theta_2 = \cdots = \theta_K = \theta$ against H_2 : at least one θ_i is different from θ, based on K independent right censored samples $X_1^{(i)}, X_2^{(i)}, \ldots, X_{n_i - s_i}^{(i)}$ $(i = 1,2,\ldots,K)$ from $E(\theta_i,\sigma)$. For simplicity of notations, let $X_i = X_1^{(i)}$ $(i = 1,2,\ldots,K)$ and $X_{(1)} = \min(X_1, X_2, \ldots, X_K)$. Then Khatri's statistics are given by

$$U_1 = \sum_{i=1}^{K} n_i(X_i - X_{(1)})/d\sigma^*,$$

$$U_2 = [\max_{1 \leq i \leq K} \{n_i(X_i - X_{(1)})\}]/d\sigma^*$$

and $$U_3 = [\max_{2 \leq i \leq K} \{n_i(X_i - X_1), n_1(X_1 - X_i)\}]/d\sigma^*.$$

He has obtained the null distributions of U_1, U_2 and U_3. Further, he has discussed their non-null distributions without carrying out any power calculations. Singh (1983) also discussed the LR procedure for testing H_o against H_2. He obtained the LR test U_4 as

$$U_4 = \sum_{i=1}^{K} n_i(X_i - X_{(1)})/\{(K-1)\sigma^*\} \equiv dU_1/(K-1)$$

and has shown that, the null distribution of U_4 is $F_{2(K-1),2d}$. However, he has not studied the power function of U_4.

It is reasonable to study a test for testing H_o against the alternative $H_1 : \theta_1 > \max_{2 \leq i \leq K} (\theta_i)$. For this purpose, in Chapter V a test statistic given by

$$T_1 = \{X_1 - \min(X_2, X_3, \ldots, X_K)\}/\sigma^*$$

is proposed. Although there are several tests (as mentioned above) for testing H_o against H_2, yet we propose another test based on

$$T_2 = \{\max_{1 \leq i \leq K} (X_i) - \min_{1 \leq i \leq K} (X_i)\}/\sigma^*.$$

Chapter V is mainly devoted for studying the performance of T_1 for K = 3 and different combinations of n_1, n_2, n_3 and d. For comparison purposes the performance of statistics U_3, U_4 and T_2 for K = 3 and $n_1 = n_2 = n_3 = n$ is also studied. Note that for equal sample sizes $U_2 = nT_2/d$. The necessary critical points of these tests are tabulated. Some exact power values of the tests T_1 and T_2 are tabulated for points in the parametric space satisfying $\theta_1 > \theta_2 > \theta_3$. Simulated power values of T_1, T_2, U_3 and U_4 are also tabulated, since the expressions for power functions of U_3 and U_4 provided by Khatri (1974) are extremely complicated. From these calculations of power values, it is concluded that

(i) the test T_1 is more sensitive to changes in n_1 compared to n_2, n_3 and d,

(ii) the power of the test T_1 is considerably higher than that of other three statistics,

(iii) the test T_2 performs slightly better than U_3 if $(\theta_1-\theta_2)$ is small, otherwise reverse is the case,

(iv) the test T_2 performs better than U_4 if $(\theta_1-\theta_2)$ is large, otherwise reverse is the case.

Finally, the LR test U_4 (or equivalently U_1) is recommended for general alternative hypothesis H_2, since its critical points are easy to evaluate from the F-distribution. For the specific alternatives like H_1, the statistic T_1 is recommended, since its critical points are available in a compact form and its power is considerably higher than that of other tests.

1.7. Generalized statistics in equal sample case, when one observation is missing on the left.

In Chapter VI, tests for the equality of location parameters of K populations are considered, when the smallest observation is missing and atleast second smallest observation is available in each sample of equal size n. Here the test defined by

$$V_1 = \{X_2^{(1)} - \min_{2 \leq i \leq K} (X_2^{(i)})\}/\sigma^*$$

is proposed for testing $H_o : \theta_1 = \theta_2 = \ldots = \theta_K$ against $H_1 : \theta_1 > \max(\theta_2, \theta_3, \ldots, \theta_K)$. Similar to the previous section, the statistics V_2 and V_3 given by

$$V_2 = \{\max_{1 \leq i \leq K} (X_2^{(i)}) - \min_{1 \leq i \leq K} (X_2^{(i)}\}/\sigma^*$$

and $$V_3 = [\max_{2 \leq i \leq K} \{(X_2^{(1)}-X_2^{(i)}), (X_2^{(i)}-X_2^{(1)})\}]/\sigma^*$$

are proposed for testing H_o against H_2 : atleast one θ_i is different from θ. Compared to two-sample case $(K = 2)$, the LR test is much more complicated even for $K = 3$. Consequently, this has not been studied at all.

For $K = 3$, the exact critical points of all the three tests are tabulated for some selected values of n and d. Since it does not appear simple to evaluate the non-null distributions of these statistics, the power of these tests are calculated by Monte-Carlo techniques for $\theta_1 > \theta_2 > \theta_3$. From this study, we have recommended V_1 for testing H_o against a specified alternative like H_1, and for testing against the alternative H_2, the test V_2 is recommended if $\theta_1 - \theta_2$ is expected to be very small, otherwise V_3 is recommended.

CHAPTER II

ESTIMATION OF THE PARAMETERS AND BASIC DISTRIBUTION THEORY

2.1. Introduction.

In this chapter, derivation and the comparison of the Least Square (LS) and Maximum Likelihood (ML) estimators of the location and the scale parameters of two-parameter exponential distributions are discussed. Some of the results are established, which are used in later chapters.

Let $X_{r+1}, X_{r+2}, \ldots, X_{n-s}$ be a type II censored sample from an exponential distribution $E(\theta, \sigma)$ with pdf

$$(2.1.1) \qquad f(x;\theta,\sigma) = \sigma^{-1} \exp\{-(x-\theta)/\sigma\}, \; 0 < \sigma < \infty, \; \theta \leq x < \infty.$$

Many researchers have investigated the problem of estimation of the scale parameter σ and the location parameter θ. For a complete sample (case $r = s = 0$) of size n, Sukhatme (1936) obtained the best unbiased estimators of θ and σ as

$$(2.1.2) \qquad \theta^* = X_1 - \sigma^*/n, \; \sigma^* = (\sum_{j=1}^{n} X_j - nX_1)/d,$$

where $d = n-r-s-1 = n-1$ for $r = s = 0$.

Lloyd (1952) discussed the technique of estimating θ and σ by applying general LS theory to an ordered sample. Type II censoring on the right (case $r = 0$) was considered by

Sarhan (1954), and he obtained the LS estimators of θ and σ as

$$(2.1.3) \quad \theta^* = X_1 - \sigma^*/n, \; \sigma^* = (\sum_{j=1}^{n-s} X_j + sX_{n-s} - nX_1)/d.$$

The LS estimators of θ and σ based on a type II doubly censored sample was discussed by Sarhan (1955), and is given by

$$(2.1.4) \quad \theta^* = X_{r+1} - \sigma^* \sum_{j=1}^{r+1} \frac{1}{n-j+1}, \; \sigma^* = \{\sum_{j=r+1}^{n-s} X_j + sX_{n-s} - (n-r)X_{r+1}\}/d.$$

Generalization for $K \geq 2$ type II doubly censored samples from $E(\theta_i, \sigma)$ are discussed in Section 2.2. The necessary distribution theory results are obtained in Section 2.3.

The ML estimators of θ and σ for a complete sample was obtained by Sukhatme (1936) as

$$(2.1.5) \quad \hat{\theta} = X_1, \; \hat{\sigma} = (\sum_{j=1}^{n} X_j - nX_1)/d^*,$$

where $d^* = n-r-s = n$ for $r = s = 0$. For type II censoring on the right, the ML estimators was discussed by Epstein and Sobel (1954). This is given by

$$(2.1.6) \quad \hat{\theta} = X_1, \; \hat{\sigma} = (\sum_{j=1}^{n-s} X_j + sX_{n-s} - nX_1)/d^*.$$

Tiku (1967) has derived the ML estimators of the parameters based on a type II doubly censored sample. He has obtained the following ML equations :

$$(2.1.7) \qquad \frac{n}{\sigma}\left[1-\frac{r}{n}-\frac{r}{n}\frac{f(z_{r+1})}{F(z_{r+1})}\right]=0$$

and

$$(2.1.8) \qquad \frac{n}{\sigma}\left[-(1-\frac{r}{n}-\frac{s}{n})+\frac{1}{n}\sum_{j=r+1}^{n-s} z_j+\frac{s}{n}z_{n-s}-\frac{r}{n}\frac{f(z_{r+1})}{F(z_{r+1})}z_{r+1}\right]=0,$$

where $f(z) = \exp(-z)$ and $F(z) = 1-\exp(-z)$ are the pdf and cdf of $Z = (X-\theta)/\sigma$ respectively. According to Tiku (1967), ML equations (2.1.7) and (2.1.8) do not have explicit solutions (for $r > 0$) due to the presence of the term $f(z)/F(z)$. He therefore obtained modified ML estimators θ_{mod} and σ_{mod} of θ and σ, which are given by

$$(2.1.9) \qquad \theta_{mod} = X_{r+1} - \sigma_{mod}\sum_{j=1}^{r+1}\frac{1}{n-j+1}$$

and $$\sigma_{mod} = \left[\sum_{j=r+1}^{n-s} X_j + sX_{n-s} - (n-r)X_{r+1}\right]/d^*.$$

In equations (2.1.7) and (2.1.8), Kambo (1978) eliminated $f(z_{r+1})/F(z_{r+1})$ and obtained explicit expressions for the ML estimators. These are given by

$$(2.1.10) \qquad \hat{\theta} = X_{r+1} + \hat{\sigma}\log(1-r/n),\ \hat{\sigma} = \sigma_{mod}.$$

Note that, for $r = 0$, equation (2.1.10) reduces to equation (2.1.6) which in turn reduces to equation (2.1.5) for $s = 0$.

From equations (2.1.4) and (2.1.10), it is easy to see that

$$(2.1.11) \qquad \theta^* = X_{r+1} - \sigma^*\sum_{j=1}^{r+1}\frac{1}{n-j+1},\ \sigma^* = \frac{d^*\hat{\sigma}}{d}.$$

In Section 2.4, ML estimators for $K(\geq 2)$ samples are derived. A brief study of MSE of ML estimators and MVU estimators is made in Section 2.5.

Epstein and Tsao (1953) considered the problem of testing equality of two exponential distributions based on right censored samples. They derived the ML estimators under the hypothesis $\theta_1 = \theta_2 = \theta$. These are given by

$$(2.1.12) \qquad \hat{\theta}_o = \min\,(X_1^{(1)}, X_1^{(2)})$$

and
$$\hat{\sigma}_o = \frac{1}{d^*} \sum_{i=1}^{2} \left[\sum_{j=1}^{n_i-s_i} (X_j^{(i)} - \hat{\theta}_o) + s_i (X_{n_i-s_i}^{(i)} - \hat{\theta}_o) \right],$$

where $d^* = \sum_{i=1}^{2} (n_i - r_i - s_i)$ with $r_1 = r_2 = 0$. For doubly censored samples, the LS estimators and the ML estimators under the hypothesis $\theta_1 = \theta_2 = \theta$ are discussed in Section 2.6.

2.2. LS estimators of the parameters for $K(\geq 2)$ samples case.

In this section, the LS estimators for K samples are derived. Let $X_{r_i+1}^{(i)}, X_{r_i+2}^{(i)}, \ldots, X_{n_i-s_i}^{(i)}$ $(i = 1, 2, \ldots, K)$ be K independent type II censored samples from $E(\theta_i, \sigma)$. Denote $E(\underset{\sim}{X}) = \underset{\sim}{A}\,\underset{\sim}{\gamma}$ and $\mathrm{Var}(\underset{\sim}{X}) = \sigma^2 \underset{\sim}{D}$,

where
$$E(X_j^{(i)}) = \theta_i + c_j^{(i)} \sigma,$$

$$\mathrm{Var}(X_j^{(i)}) = a_j^{(i)} \sigma^2,$$

$$c_j^{(i)} = \sum_{g=1}^{j} 1/(n_i - g + 1),$$

$$a_j^{(i)} = \sum_{g=1}^{j} 1/(n_i - g + 1)^2,$$

$$\underset{\sim}{X} = \begin{bmatrix} \underset{\sim}{X}(1) \\ \underset{\sim}{X}(2) \\ \vdots \\ \underset{\sim}{X}(K) \end{bmatrix}, \quad \underset{\sim}{X}(i) = \begin{bmatrix} X^{(i)}_{r_i+1} \\ X^{(i)}_{r_i+2} \\ \vdots \\ X^{(i)}_{n_i-s_i} \end{bmatrix}, \quad \underset{\sim}{\gamma} = \begin{bmatrix} \underset{\sim}{\theta} \\ \sigma \end{bmatrix}, \quad \underset{\sim}{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_K \end{bmatrix},$$

$$\underset{\sim}{A} = \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{0} & \cdots & \underset{\sim}{0} & \underset{\sim}{C}(1) \\ \underset{\sim}{0} & \underset{\sim}{1} & \cdots & \underset{\sim}{0} & \underset{\sim}{C}(2) \\ \vdots & & & & \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{1} & \underset{\sim}{C}(K) \end{bmatrix}_{d^* \times (K+1)}, \quad \underset{\sim}{C}(i) = \begin{bmatrix} c^{(i)}_{r_i+1} \\ c^{(i)}_{r_i+2} \\ \vdots \\ c^{(i)}_{n_i-s_i} \end{bmatrix},$$

and $\text{rank}(\underset{\sim}{A}) = K+1$; $\underset{\sim}{D}$ is the dispersion matrix of $\underset{\sim}{X}/\sigma$, and is given by

$$\underset{\sim}{D} = \begin{bmatrix} \underset{\sim}{D}_1 & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{D}_2 & \cdots & \underset{\sim}{0} \\ \vdots & & & \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{D}_K \end{bmatrix}_{d^* \times d^*},$$

$$\underset{\sim}{D}_i = \begin{bmatrix} a^{(i)}_{r_i+1} & a^{(i)}_{r_i+1} & \cdots & a^{(i)}_{r_i+1} \\ a^{(i)}_{r_i+1} & a^{(i)}_{r_i+2} & \cdots & a^{(i)}_{r_i+2} \\ \vdots & & & \\ a^{(i)}_{r_i+1} & a^{(i)}_{r_i+2} & \cdots & a^{(i)}_{n_i-s_i} \end{bmatrix}_{d_i^* \times d_i^*}$$

$d^* = \sum_{i=1}^{K} d_i^*$, $d_i^* = n_i - r_i - s_i$, $j = r_i+1, r_i+2, \ldots, n_i-s_i$; $i=1,2,\ldots,K$;

and $\underset{\sim}{1}$ and $\underset{\sim}{0}$ are the matrices of suitable orders with all entries as 1 and 0 respectively.

Following Lloyd (1952), the LS estimator of $\underset{\sim}{\gamma}$ in this setup is given by

$$(2.2.1) \qquad \underset{\sim}{\gamma} = \begin{bmatrix} \underset{\sim}{\theta}^* \\ \sigma^* \end{bmatrix} = (\underset{\sim}{A}' \underset{\sim}{\Omega} \underset{\sim}{A})^{-1} (\underset{\sim}{A}' \underset{\sim}{\Omega} \underset{\sim}{X}),$$

where $$\underset{\sim}{\Omega} = \underset{\sim}{D}^{-1} = \begin{bmatrix} \underset{\sim}{\Omega}_1 & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ 0 & \underset{\sim}{\Omega}_2 & \cdots & \underset{\sim}{0} \\ \vdots & & & \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{\Omega}_K \end{bmatrix} \text{ and } \underset{\sim}{\Omega}_i = \underset{\sim}{D}_i^{-1}.$$

That is,

$$\underset{\sim}{\Omega}_i = \begin{bmatrix} \frac{1}{a_i} + (n_i - r_i - 1)^2 & -(n_i - r_i - 1)^2 & 0 & \cdots & 0 \\ -(n_i - r_i - 1)^2 & (n_i - r_i - 1)^2 + (n_i - r_i - 2)^2 & -(n_i - r_i - 2)^2 & \cdots & 0 \\ \vdots & & & & -(s_i+1)^2 \\ 0 & 0 & 0 & \cdots & (s_i+1)^2 \end{bmatrix}$$

$$\text{Hence, } \underset{\sim}{A}' \underset{\sim}{\Omega} \underset{\sim}{A} = \left[\begin{array}{cccc|c} 1/a_1 & 0 & \cdots & 0 & b_1/a_1 \\ 0 & 1/a_2 & \cdots & 0 & b_2/a_2 \\ \vdots & & & & \\ 0 & 0 & \cdots & 1/a_K & b_K/a_K \\ \hline \frac{b_1}{a_1} & \frac{b_2}{a_2} & \cdots & \frac{b_K}{a_K} & Q \end{array}\right]_{(K+1)\times(K+1)},$$

where $$b_i = c^{(i)}_{r_i+1} = \sum_{j=1}^{r_i+1} (n_i - j + 1)^{-1},$$

$$a_i = a^{(i)}_{r_i+1} = \sum_{j=1}^{r_i+1} (n_i - j + 1)^{-2},$$

$$Q = d + \sum_{i=1}^{K} (b_i^2/a_i), \quad d = \sum_{i=1}^{K} d_i \text{ and } d_i = d_i^* - 1 \text{ for } i = 1, 2, \ldots K.$$

Now, for a partioned matrix $\begin{bmatrix} \underset{\sim}{R} & \underset{\sim}{S} \\ \underset{\sim}{S}' & \underset{\sim}{T} \end{bmatrix}$ (where $\underset{\sim}{R}$ is non-singular), the inverse is (see, for example Rao 1973, p. 33)

$$\begin{bmatrix} \underset{\sim}{R}^{-1} + \underset{\sim}{F}\,\underset{\sim}{E}^{-1}\,\underset{\sim}{F}' & -\underset{\sim}{F}\,\underset{\sim}{E}^{-1} \\ -\underset{\sim}{E}^{-1}\,\underset{\sim}{F}' & \underset{\sim}{E}^{-1} \end{bmatrix},$$

where $\underset{\sim}{E} = \underset{\sim}{T} - \underset{\sim}{S}'\,\underset{\sim}{R}^{-1}\,\underset{\sim}{S}$ and $\underset{\sim}{F} = \underset{\sim}{R}^{-1}\,\underset{\sim}{S}$.

Applying this we have

$$(2.2.2)\qquad (\underset{\sim}{A}'\,\underset{\sim}{\Omega}\,\underset{\sim}{A})^{-1} = \frac{1}{d}\left[\begin{array}{cccc|c} a_1d+b_1^2 & b_1b_2 & \cdots & b_1b_K & -b_1 \\ b_1b_2 & a_2d+b_2^2 & \cdots & b_2b_K & -b_2 \\ \vdots & \vdots & & & \vdots \\ b_1b_K & b_2b_K & \cdots & a_Kd+b_K^2 & -b_K \\ \hline -b_1 & -b_2 & \cdots & -b_K & 1 \end{array}\right].$$

Now,

$$(\underset{\sim}{A}'\,\underset{\sim}{\Omega}\,\underset{\sim}{X}) = \begin{bmatrix} X^{(1)}_{r_1+1}/a_1 \\ X^{(2)}_{r_2+1}/a_2 \\ \vdots \\ X^{(K)}_{r_K+1}/a_K \\ P_1 + \sum\limits_{i=1}^{K} b_i X^{(i)}_{r_i+1}/a_i \end{bmatrix},$$

where $P_1 = \sum\limits_{i=1}^{K} \left[\sum\limits_{j=r_i+1}^{n_i-s_i} X^{(i)}_j + s_i X^{(i)}_{n_i-s_i} - (n_i-r_i) X^{(i)}_{r_i+1} \right]$. Substituting for $(\underset{\sim}{A}'\,\underset{\sim}{\Omega}\,\underset{\sim}{A})^{-1}$ and $(\underset{\sim}{A}'\,\underset{\sim}{\Omega}\,\underset{\sim}{X})$ in equation (2.2.1) we obtain for $i = 1, 2, \ldots, K$

$$(2.2.3)\qquad \theta^*_i = X^{(i)}_{r_i+1} - b_i\sigma^*,\quad \sigma^* = P_1/d\ ,$$

where

(2.2.4)
$$b_i = \sum_{j=1}^{r_i+1} (n_i-j+1)^{-1} .$$

Note that for K = 1, equation (2.2.3) reduces to equation (2.1.4) obtained by Sarhan (1955).

The variance covariance matrix of $\underset{\sim}{\gamma}^*$ is $(\underset{\sim}{A}' \underset{\sim}{Q} \underset{\sim}{A})^{-1}\sigma^2$. From equation (2.2.2) we have

(2.2.5) $\quad \mathrm{Var}(\theta_i^*) = [a_i+b_i^2/d] \;\sigma^2 \quad (i = 1, 2, \ldots, K),$

(2.2.6) $\quad \mathrm{Var}(\sigma^*) = \sigma^2/d$

and

(2.2.7) $\mathrm{Covar}(\theta_i^*,\sigma^*) = -b_i\sigma^2/d, \mathrm{Covar}(\theta_i^*,\theta_j^*)=b_ib_j\sigma^2/d\,(i\neq j=1, 2, \ldots, K).$

Hence,

(2.2.8) $\quad \mathrm{Var}(\theta_1^*-\theta_2^*) = [a_1+a_2+(b_1-b_2)^2/d\,]\sigma^2.$

2.3. Distribution theory.

In this section, the distribution of the statistics are discussed, which are main tool for deriving the distribution of the LS estimators.

Theorem 2.3.1. $X^{(1)}_{r_1+1}, X^{(2)}_{r_2+1}, \ldots, X^{(K)}_{r_K+1}$ and σ^* are independently distributed.

Proof. The jpdf of $\underset{\sim}{X}(1), \underset{\sim}{X}(2), \ldots, \underset{\sim}{X}(K)$ is

$$f(\underset{\sim}{x}(1), \underset{\sim}{x}(2), \ldots, \underset{\sim}{x}(K); \theta_1, \theta_2, \ldots, \theta_K, \sigma)$$
$$= \prod_{i=1}^{K} \frac{n_i! \,[\,1-\exp\{-(x^{(i)}_{r_i+1}-\theta_i)/\sigma\}]^{r_i}}{r_i!\; s_i!\; \sigma^{n_i-r_i-s_i}} \times$$

where

$$(2.2.4) \qquad b_i = \sum_{j=1}^{r_i+1} (n_i-j+1)^{-1} .$$

Note that for K = 1, equation (2.2.3) reduces to equation (2.1.4) obtained by Sarhan (1955).

The variance covariance matrix of $\underset{\sim}{\gamma}^*$ is $(\underset{\sim}{A}'\underset{\sim}{Q}\,\underset{\sim}{A})^{-1}\sigma^2$. From equation (2.2.2) we have

$$(2.2.5) \qquad \mathrm{Var}(\theta_i^*) = [a_i+b_i^2/d]\ \sigma^2 \quad (i = 1,2,\ldots,K),$$

$$(2.2.6) \qquad \mathrm{Var}(\sigma^*) = \sigma^2/d$$

and

$$(2.2.7)\ \mathrm{Covar}(\theta_i^*,\sigma^*) = -b_i\sigma^2/d, \mathrm{Covar}(\theta_i^*,\theta_j^*)=b_ib_j\sigma^2/d\,(i\neq j=1,2,\ldots,K).$$

Hence,

$$(2.2.8) \qquad \mathrm{Var}(\theta_1^*-\theta_2^*) = [a_1+a_2+(b_1-b_2)^2/d\,]\sigma^2.$$

2.3. Distribution theory.

In this section, the distribution of the statistics are discussed, which are main tool for deriving the distribution of the LS estimators.

Theorem 2.3.1. $X^{(1)}_{r_1+1}, X^{(2)}_{r_2+1}, \ldots, X^{(K)}_{r_K+1}$ and σ^* are independently distributed.

Proof. The jpdf of $\underset{\sim}{X}(1),\underset{\sim}{X}(2),\ldots,\underset{\sim}{X}(K)$ is

$$f(\underset{\sim}{x}(1),\underset{\sim}{x}(2),\ldots,\underset{\sim}{x}(K);\theta_1,\theta_2,\ldots,\theta_K,\sigma)$$

$$= \prod_{i=1}^{K} \frac{n_i!\,[1-\exp\{-(x^{(i)}_{r_i+1}-\theta_i)/\sigma\}]^{r_i}}{r_i!\,s_i!\,\sigma^{n_i-r_i-s_i}} \times$$

$$\cdot \exp\{-s_i(x^{(i)}_{n_i-s_i}-\theta_i)/\sigma - \sum_{j=r_i+1}^{n_i-s_i}(x_j^{(i)}-\theta_i)/\sigma\}$$

$$\text{for } x^{(i)}_{r_i+1} \geq \theta_i \quad (i = 1,2,\ldots,K).$$

For the ith sample $(i = 1,2,\ldots,K)$, considering the transformations,

$$(2.3.1) \quad \begin{cases} y^{(i)}_{r_i+1} = (n_i-r_i)(x^{(i)}_{r_i+1}-\theta_i)/\sigma, \\ \\ y^{(i)}_j = (n_i-j+1)(x^{(i)}_j-x^{(i)}_{j-1})/\sigma \quad (j = r_i+2,\ldots,n_i-s_i) \end{cases}$$

and noting that $y^{(i)}_j > 0$, the corresponding inverse transformation is

$$x^{(i)}_{r_i+1} = \sigma\, y^{(i)}_{r_i+1}/(n_i-r_i) + \theta_i,$$

$$x^{(i)}_{r_i+2} = \sigma\, y^{(i)}_{r_i+1}/(n_i-r_i) + \sigma\, y^{(i)}_{r_i+2}/(n_i-r_i-1)+\theta_i,$$

$$\vdots$$

$$x^{(i)}_{n_i-s_i} = \sigma\, y^{(i)}_{r_i+1}/(n_i-r_i)+\sigma\, y^{(i)}_{r_i+2}/(n_i-r_i-1)+\cdots$$

$$+\sigma\, y^{(i)}_{n_i-s_i}/(s_i+1)+\theta_i.$$

Hence,

$$\sigma \sum_{j=r_i+1}^{n_i-s_i} y^{(i)}_j = \sum_{j=r_i+1}^{n_i-s_i}(x^{(i)}_j-\theta_i) + s_i(x^{(i)}_{n_i-s_i}-\theta_i).$$

Following standard methods, the jacobian of the transformation is

$$J = \prod_{i=1}^{K} \frac{s_i!\ \sigma^{n_i-r_i-s_i}}{(n_i-r_i)!}.$$

The jpdf of $Y_j^{(i)}$'s ($j = r_i+1, \ldots, n_i-s_i$; $i = 1, 2, \ldots, K$) is then obtained as

$$f(\underset{\sim}{y}(1), \underset{\sim}{y}(2) \ldots, \underset{\sim}{y}(K))$$

$$= \prod_{i=1}^{K} \frac{n_i!}{r_i!\,(n_i-r_i)!} [1-\exp\{-y_{r_i+1}^{(i)}/(n_i-r_i)\}]^{r_i} \exp\{-\sum_{j=r_i+1}^{n_i-s_i} y_j^{(i)}\}$$

$$(2.3.2) \quad = \prod_{i=1}^{K} \left[\frac{n_i!}{r_i!(n_i-r_i)!} [1-\exp\{-y_{r_i+1}^{(i)}/(n_i-r_i)\}]^{r_i} \exp\{-y_{r_i+1}^{(i)}\}\right]$$

$$\cdot \prod_{j=r_i+2}^{n_i-s_i} [\exp\{-y_j^{(i)}\}] \quad \text{for } y_j^{(i)} \geq 0.$$

This shows that $Y_j^{(i)}$'s are independently distributed with the following density functions :

$$(2.3.3) \quad f_{Y_{r_i+1}^{(i)}}(y) = \frac{n_i!}{r_i!(n_i-r_i)!} [1-\exp\{-y/(n_i-r_i)\}]^{r_i} \exp(-y), y \geq 0$$

and

$$(2.3.4) \quad f_{Y_j^{(i)}}(y) = \exp(-y), \ y \geq 0$$

for $j = r_i+2, r_i+3, \ldots, n_i-s_i$ and $i = 1, 2, \ldots, K$.

Note that,

$$(2.3.5) \quad \sum_{i=1}^{K} \sum_{j=r_i+2}^{n_i-s_i} y_j^{(i)} = d\sigma^*/\sigma,$$

which is free from $y_{r_i+1}^{(i)}$ ($i = 1, 2, \ldots, K$).

From equations (2.3.2) to (2.3.5) and transformation (2.3.1), it is easy to see that $X_{r_1+1}^{(1)}, X_{r_2+1}^{(2)}, \ldots, X_{r_K+1}^{(K)}$ and σ^* are independently distributed. This completes the proof of the theorem.

<u>Corollary 2.3.1.</u> $2d\sigma^*/\sigma$ has a chi-square distribution with 2d degrees of freedom (DF).

<u>Proof.</u> By making the transformation $z_j^{(i)} = 2y_j^{(i)}(j=r_i+2, \ldots, n_i-s_i;$ $i = 1, 2, \ldots, K)$ in equation (2.3.3), we see that $z_j^{(i)}$'s are

iid chi-square variates with 2 DF. Consequently,

$$\sum_{i=1}^{K} \sum_{j=r_i+2}^{n_i-s_i} z_j^{(i)} = 2d\sigma^*/\sigma$$

has a chi-square distribution with 2d DF.

In particular, the pdf of $W = d\sigma^*/\sigma$ is given by

$$(2.3.6) \qquad f(w) = w^{d-1}\, e^{-w}/\Gamma(d), \quad w \geq 0.$$

<u>Corollary 2.3.2.</u> The pdf of $X_{r_i+1}^{(i)}$ $(i = 1, 2, \ldots, K)$ is given by

$$(2.3.7) \quad f_{X_{r_i+1}^{(i)}}(x) = \frac{[1-\exp\{-(x-\theta_i)/\sigma\}]^{r_i}}{B(r_i+1, n_i-r_i)\sigma} \exp\{-(n_i-r_i)(x-\theta_i)/\sigma\}$$

for $x \geq \theta_i$, where $B(r_i+1, n_i-r_i) = r_i!\,(n_i-r_i-1)!/n_i!$.

<u>Proof.</u> The proof follows directly from the marginal pdf of $X_{r_i+1}^{(i)}$ or from equation (2.3.3) and the transformation given by equation (2.3.1).

The distribution of the LS estimator θ_i^* of $\theta_i (i=1, 2, \ldots, K)$ is given in the following corollary:

<u>Corollary 2.3.3.</u> The pdf of $Y_i = (\theta_i^*-\theta_i)/\sigma$ $(i = 1, 2, \ldots, K)$ is

$$(2.3.8) \quad f_{Y_i}(y) = \begin{cases} p_1(y), & y < 0 \\ p_2(y), & y \geq 0, \end{cases}$$

where

$$p_1(y) = G \sum_{j=0}^{r_i} (-1)^j \binom{r_i}{j} \exp\{dy/b_i\} \sum_{h=0}^{d-1} \frac{\{-(n_i-r_i+j+d/b_i)y\}^h}{h!\,(n_i-r_i+j+d/b_i)^d},$$

$$p_2(y) = G \sum_{j=0}^{r_i} (-1)^j \binom{r_i}{j} \exp\{-(n_i-r_i+j)y\}/(n_i-r_i+j+d/b_i)^d,$$

$$G = (d/b_i)^d/B(r_i+1, n_i-r_i), \quad b_i = \sum_{j=1}^{r_i+1} (n_i-j+1)^{-1}$$

and θ_i^* is given in equation (2.2.3).

Proof. The proof follows immediately on making suitable transformation and using Theorem 2.3.1 along with equations (2.3.6) and (2.3.7).

2.4. ML estimators of the parameters for $K(\geq 2)$ samples case.

Let $\underset{\sim}{X}(i)$ $(i = 1, 2, \ldots, K)$ be $(K \geq 2)$ independent type II doubly censored samples from $E(\theta_i, \sigma)$. Then the likelihood function is

$$L(\theta_1, \theta_2, \ldots, \theta_K, \sigma \mid \underset{\sim}{x}(1), \underset{\sim}{x}(2), \ldots, \underset{\sim}{x}(K))$$

$$= \prod_{i=1}^{K} \frac{n_i!\,\sigma^{-d_i^*}}{r_i!\;s_i!}[1-e^{-(x_{r_i+1}^{(i)}-\theta_i)/\sigma}]^{r_i} \exp[-\tfrac{1}{\sigma}\{s_i(x_{n_i-s_i}^{(i)}-\theta_i)$$

$$+ \sum_{j=r_i+1}^{n_i-s_i} (x_j^{(i)}-\theta_i)\}] \text{ for } x_{r_i+1}^{(i)} \geq \theta_i (i=1, 2, \ldots, K), \sigma > 0,$$

where $d_i^* = n_i-r_i-s_i$. For $i = 1, 2, \ldots, K$ make the substitution

$$z_j^{(i)} = (x_j^{(i)}-\theta_i)/\sigma \quad (j = r_i+1, \ldots, n_i-s_i),$$

then we have

$$L(\theta_1,\theta_2,\ldots,\theta_K,\sigma \mid \underset{\sim}{z}(1),\underset{\sim}{z}(2),\ldots,\underset{\sim}{z}(K))$$

$$= \prod_{i=1}^{K} \frac{n_i!\,\sigma^{-d_i^*}}{r_i!\,s_i!}\,[1-e^{-z_{r_i+1}^{(i)}}]^{r_i} \exp\,[-s_i z_{n_i-s_i}^{(i)} - \sum_{j=r_i+1}^{n_i-s_i} z_j^{(i)}]$$

$$\text{for } z_{r_i+1}^{(i)} \geq 0 \;(i = 1,2,\ldots,K).$$

Differentiating $\log L(\theta_1,\theta_2,\ldots,\theta_K,\sigma \mid \underset{\sim}{z}(1),\underset{\sim}{z}(2),\ldots,\underset{\sim}{z}(K))$ w.r.to θ_i $(i = 1,2,\ldots,K)$ and σ we get the following $(K+1)$ likelihood equations :

$$(2.4.1)\qquad r_i/[e^{z_{r_i+1}^{(i)}}-1] - (n_i-r_i) = 0 \quad (i = 1,2,\ldots,K)$$

and

$$(2.4.2)\qquad \sum_{i=1}^{K} [\sum_{j=r_i+1}^{n_i-s_i} z_j^{(i)} + s_i z_{n_i-s_i}^{(i)} - r_i z_{r_i+1}^{(i)}/\{e^{z_{r_i+1}^{(i)}}-1\}] - d^* = 0.$$

Equation (2.4.1) simplifies to

$$\hat{\theta}_i = x_{r_i+1}^{(i)} + \hat{\sigma}\log(1-r_i/n_i)$$

$$(2.4.3)\qquad = x_{r_i+1}^{(i)} - \hat{\sigma}\, m_i,$$

where $m_i = \log\{n_i/(n_i-r_i)\}$ $(i = 1,2,\ldots,K)$.

Substitution of (2.4.1) in equation (2.4.2) gives

$$(2.4.4)\qquad \hat{\sigma} = d\sigma^*/d^* = P_1/d^*,$$

where σ^* is given in equation (2.2.3).

The distributions of $\hat{\sigma}$ and $\hat{\theta}_i$ can be easily obtained by applying Theorem 2.3.1 and its corollaries.

2.5. Comparison of ML and LS estimators.

Epstein and Sobel (1954) showed that, the LS estimators given in equation (2.1.3) are the minimum variance unbiased (MVU) estimators of θ and σ respectively. It follows that θ_i^* $(i = 1, 2, \ldots, K)$ and σ^* given in equation (2.2.3) are the MVU estimators of θ_i $(i = 1, 2, \ldots, K)$ and σ respectively [see, Sarhan and Greenberg 1962, p. 368].

Then from equations (2.2.3) and (2.4.4) we have

$$(2.5.1) \qquad \sigma^* = d^*\hat{\sigma}/d\ , \quad \theta_i^* = X_{r_i+1}^{(i)} - b_i\sigma^*.$$

Using the equations (2.2.5), (2.2.6) and (2.5.1), the means, variances and mean square errors (MSE) of ML estimator $\hat{\sigma}$ are respectively

$$(2.5.2) \qquad E(\hat{\sigma}) = d\sigma/d^*$$

$$(2.5.3) \qquad \mathrm{Var}(\hat{\sigma}) = d\sigma^2/d^{*2}$$

$$(2.5.4) \qquad \mathrm{MSE}(\hat{\sigma}) = (d+K^2)\sigma^2/(d+K)^2$$

From relations (2.2.5) and (2.5.3), it can be seen that

$$\mathrm{Var}(\hat{\sigma}) < \mathrm{Var}(\sigma^*).$$

The relative efficiency E of $\hat{\sigma}$ w.r.to σ^* is given by

$$E = \mathrm{MSE}(\sigma^*)/\mathrm{MSE}(\hat{\sigma}) = (d+K)^2/(d^2+dK^2).$$

Note that, E is less than 1 whenever $d > K/(K-2)$. Since in general d is large, E is less than 1 for most values of $K \geq 3$.

This shows that in general for $K \geq 3$, σ^* is a better estimator than $\hat{\sigma}$. However, for $K \leq 2$, the value of E is greater than 1 and the ML estimator $\hat{\sigma}$ even though biased, is better than σ^*. This agrees with the conclusions drawn by Kambo (1978) for $K = 1$. Further, for fixed K, E tends to 1 as d tends to infinity. Table 2.5.1 gives E for $K = 1(1)6(2)10$ and $d = 2(2)10(5)30$. This table shows that for $K \geq 3$ and moderate values of d, the estimator σ^* is considerably better than $\hat{\sigma}$.

The equation (2.4.3) also gives

$$E(\hat{\theta}_i) = \theta_i + \sigma(b_i - A_i) \text{ and } Var(\hat{\theta}_i) = \{a_i + A_i^2/d\}\sigma^2.$$

Hence,

$$\text{(2.5.5)} \qquad MSE(\hat{\theta}_i) = \{(a_i + A_i^2/d) + (b_i - A_i)^2\} \sigma^2,$$

where $A_i = dm_i/d^*$ $(i = 1, 2, \ldots, K)$.

In the general case direct comparison of the MSE of $\hat{\theta}_i$ and θ_i^* is difficult. But from equations (2.5.5) and (2.2.5), it is clear that, for $r_i = 0$ $(i = 1, 2, \ldots, K)$ $MSE(\theta_i^*) < MSE(\hat{\theta}_i)$. For the two samples case, $n_1 = 5$, $r_1 = 0$, $s_1 = 2$, $n_2 = 15$, $r_2 = 12$ and $s_2 = 1$, the MSE of estimators are calculated from equations (2.2.5) and (2.5.5), and are given by

$$MSE(\theta_1^*) = 0.0533\sigma^2 < MSE(\hat{\theta}_1) = 0.800\sigma^2$$

and

$$MSE(\theta_2^*) = 1.4324\sigma^2 > MSE(\hat{\theta}_2) = 1.3681\sigma^2.$$

This shows that nothing can be concluded about the relative efficiencies of these estimators. The proper estimator out of these two is the one with the smaller MSE. Similar conclusions are drawn by Kambo (1978) for the case of one population.

For later use, we now evaluate the variance of $(\hat{\theta}_1-\hat{\theta}_2)$. From equations (2.4.3) and (2.4.4), we have

$$\begin{aligned}\hat{\theta}_1-\hat{\theta}_2 &= (X^{(1)}_{r_1+1}-X^{(2)}_{r_2+1}) - \hat{\sigma}(m_1-m_2)\\ &= (X^{(1)}_{r_1+1}-X^{(2)}_{r_2+1}) - md\sigma^*/d^*,\end{aligned}$$

where $m = m_1-m_2$. From Theorem 2.3.1 and equation (2.2.6) we get

$$\text{(2.5.6)} \qquad \mathrm{Var}(\hat{\theta}_1-\hat{\theta}_2) = \{a_1+a_2+m^2d/d^{*2}\}\sigma^2.$$

2.6. Estimators of the parameters under the hypothesis $\theta_1=\theta_2=\theta$.

The LS estimators and the ML estimators of the parameters θ and σ under the hypothesis $\theta_1 = \theta_2 = \theta$, based on two independent type II censored samples are discussed in this section.

2.6.1. LS estimators of the parameters. In this subsection, all unspecified notations are as given in Section 2.2 with $K = 2$. Denote $E(\underset{\sim}{X}) = \underset{\sim}{B}\,\underset{\sim}{\gamma}$ and $\mathrm{Var}(\underset{\sim}{X}) = \sigma^2\underset{\sim}{D}$,

where $$\underset{\sim}{B} = \begin{bmatrix} \underset{\sim}{1} & \underset{\sim}{C}(1) \\ \underset{\sim}{1} & \underset{\sim}{C}(2) \end{bmatrix} \text{ and } \underset{\sim}{\gamma} = \begin{bmatrix} \theta \\ \sigma \end{bmatrix}.$$

Then the LS estimator of $\underset{\sim}{\gamma}$ is given by

$$(2.6.1) \qquad \underset{\sim}{\gamma}_o^* = \begin{bmatrix} \theta_o^* \\ \sigma_o^* \end{bmatrix} = (\underset{\sim}{B}' \underset{\sim}{\Omega} \underset{\sim}{B})^{-1} (\underset{\sim}{B}' \underset{\sim}{\Omega} \underset{\sim}{X}),$$

where $(\underset{\sim}{B}' \underset{\sim}{\Omega} \underset{\sim}{B}) = \begin{bmatrix} 1/a_1 + 1/a_2 & b_1/a_1 + b_2/a_2 \\ b_1/a_1 + b_2/a_2 & Q \end{bmatrix}$,

$$(2.6.2) \quad (\underset{\sim}{B}' \underset{\sim}{\Omega} \underset{\sim}{B})^{-1} = \frac{1}{Q^*} \begin{bmatrix} Q & -(b_1/a_1+b_2/a_2) \\ -(b_1/a_1+b_2/a_2) & 1/a_1+1/a_2 \end{bmatrix},$$

where $Q^* = |\underset{\sim}{B}' \underset{\sim}{\Omega} \underset{\sim}{B}| = \{(b_1-b_2)^2 + d(a_1+a_2)\}/a_1a_2.$

Further,

$$(\underset{\sim}{B}' \underset{\sim}{\Omega} \underset{\sim}{X}) = \begin{bmatrix} x^{(1)}_{r_1+1}/a_1 + x^{(2)}_{r_2+1}/a_2 \\ P_1 + \sum_{i=1}^{2} b_i \, x^{(i)}_{r_i+1}/a_i \end{bmatrix}.$$

Simplification of equation (2.6.1) gives

$$(2.6.3) \quad \theta_o^* = \frac{1}{Q^*} \left[Q\left\{\frac{x^{(1)}_{r_1+1}}{a_1} + \frac{x^{(2)}_{r_2+1}}{a_2}\right\} - \left(\frac{b_1}{a_1} + \frac{b_2}{a_2}\right)\left\{P_1 + \sum_{i=1}^{2} \frac{b_i x^{(i)}_{r_i+1}}{a_i}\right\}\right]$$

and

$$(2.6.4) \quad \sigma_o^* = \frac{1}{Q^*} \left[-\left(\frac{b_1}{a_1} + \frac{b_2}{a_2}\right)\left\{\frac{x^{(1)}_{r_1+1}}{a_1} + \frac{x^{(2)}_{r_2+1}}{a_2}\right\} + \left(\frac{1}{a_2} + \frac{1}{a_1}\right)\left\{P_1 + \sum_{i=1}^{2} b_i x^{(i)}_{r_i+1}/a_i\right\}\right].$$

Substituting for $(P_1 + \sum_{i=1}^{2} b_i x^{(i)}_{r_i+1}/a_i)$ in equation (2.6.3) by (2.6.4), and simplifying, we get the unbiased LS estimators of θ and σ as

$$(2.6.5) \quad \theta_o^* = \{a_2X^{(1)}_{r_1+1}+a_1X^{(2)}_{r_2+1} - \sigma_o^*(b_1a_2+b_2a_1)\}/(a_1+a_2)$$

and

$$(2.6.6) \quad \sigma_o^* = \{(b_1-b_2)\{X^{(1)}_{r_1+1}-X^{(2)}_{r_2+1}\}+d(a_1+a_2)\sigma^*\}/\{d(a_1+a_2)+(b_1-b_2)^2\}.$$

The variance covariance matrix of $\underset{\sim}{\gamma}_o^*$ is $(\underset{\sim}{B}'\underset{\sim}{\Omega}\underset{\sim}{B})^{-1}\sigma^2$. The equation (2.6.2) now gives

$$(2.6.7) \quad \mathrm{Var}(\theta_o^*) = \Omega\sigma^2/\Omega^*, \ \mathrm{Var}(\sigma_o^*) = (a_1+a_2)\sigma^2/(a_1a_2\Omega^*)$$

and

$$\mathrm{Covar}(\theta_o^*,\sigma_o^*) = -(a_1b_2+a_2b_1)\sigma^2/(a_1a_2\Omega^*).$$

2.6.2. <u>ML estimators of the parameters.</u>

The likelihood function of $\underset{\sim}{X}(1)$ and $\underset{\sim}{X}(2)$ is given by

$$L(\theta,\sigma\,|\,\underset{\sim}{x}(1),\underset{\sim}{x}(2)) = \prod_{i=1}^{2} \frac{n_i!\,\sigma^{-d_i^*}}{r_i!\,s_i!}\,[1-\exp\{x^{(i)}_{r_i+1}-\theta\}/\sigma]^{r_i}$$

$$\cdot \exp\Big[-\frac{1}{\sigma}\{s_i(x^{(i)}_{n_i-s_i}-\theta)+\sum_{j=r_i+1}^{n_i-s_i}(x_j^{(i)}-\theta)\}\Big], x^{(i)}_{r_i+1} \geq \theta\,(i=1,2), \sigma > 0.$$

Without loss of generality, let $x^{(1)}_{r_1+1} \leq x^{(2)}_{r_2+1}$, for otherwise, we can simply relabel the samples. Now $y = x^{(2)}_{r_2+1}-x^{(1)}_{r_1+1} \geq 0$ and

$$(2.6.8) \quad L(\theta,\sigma\,|\,\underset{\sim}{x}(1),\underset{\sim}{x}(2)) = \mathrm{Const}.\sigma^{-d^*}[1-\exp\{-(x^{(1)}_{r_1+1}-\theta)/\sigma\}]^{r_1}$$

$$\cdot\,[1-\exp\{-\frac{y}{\sigma}-(x^{(1)}_{r_1+1}-\theta)/\sigma\}]^{r_2}\exp\Big[-\frac{1}{\sigma}\{P+f(x^{(1)}_{r_1+1}-\theta)\}\Big]$$

$$\text{for } x^{(1)}_{r_1+1} \geq \theta, \ \sigma > 0,$$

where $P = \sum_{i=1}^{2} \left[\sum_{j=r_i+1}^{n_i-s_i} x_j^{(i)} + s_i x_{n_i-s_i}^{(i)} - (n_i-r_i) x_{r_1+1}^{(1)} \right]$ and $f=n_1+n_2-r_1-r_2$.

According to zero or non-zero values of r_1 and r_2, we have four different cases.

Case (I): $r_1, r_2=0$. The likelihood function (2.6.8) becomes,

$$L(\theta,\sigma \mid \underset{\sim}{x}(1),\underset{\sim}{x}(2)) = \text{Const.}\ \exp\left[-\frac{1}{\sigma}\{P+f(x_1^{(1)}-\theta)\}\right]/\sigma^{d^*}$$

$$\text{for } x_1^{(1)} > \theta,\ \sigma > 0.$$

It is clear that L is maximum for $\hat{\theta}_o = X_1^{(1)}$ and $\hat{\sigma}_o = P/d^*$. This has been also obtained by Epstein and Tsao (1953).

Case (II) : $r_1 > 0$, $r_2 = 0$. Now, the likelihood function (2.6.8) simplifies to

$$(2.6.9) \qquad L(\theta,\sigma \mid \underset{\sim}{z}(1),\underset{\sim}{z}(2))$$
$$= \frac{\text{Const.}}{\sigma^{d^*}}\left[1-e^{-z_{r_1+1}^{(1)}}\right]^{r_1} \exp\left[-\sum_{i=1}^{2}\{s_i z_{n_i-s_i}^{(i)} + \sum_{j=r_i+1}^{n_i-s_i} z_j^{(i)}\}\right]$$

$$\text{for } z_{r_1+1}^{(1)} \geq 0,\ \sigma > 0,$$

where $z_j^{(i)} = (x_j^{(i)}-\theta)/\sigma$, $j = r_i+1,\ldots,n_i-s_i$ and $i = 1,2$.

The maximizing equations for $L(\theta,\sigma \mid \underset{\sim}{z}(1),\underset{\sim}{z}(2))$ are

$$(2.6.10) \qquad r_1/[\exp\{z_{r_1+1}^{(1)}\}-1] - f = 0$$

and

$$(2.6.11) \qquad r_1 z_{r_1+1}^{(1)}/[\exp\{z_{r_1+1}^{(1)}\}-1] - f z_{r_1+1}^{(1)} - P/\sigma + d^* = 0.$$

Solving the equations (2.6.10) and (2.6.11), we get

(2.6.12) $\hat{\theta}_o = X^{(1)}_{r_1+1} - \hat{\sigma}_o \log (1+r_1/f)$, $\hat{\sigma}_o = P/d^*$.

Note that $\hat{\theta}_o \leq X^{(1)}_{r_1+1}$.

Case (III) : $r_1 = 0$, $r_2 > 0$. In this case the likelihood function (2.6.8) reduces to

$$(2.6.13) \quad L(\theta,\sigma \mid \underset{\sim}{x}(1), \underset{\sim}{x}(2)) = \text{Const.}\ \sigma^{-d^*} \{1-qw\}^{r_2} e^{-P/\sigma} w^f$$

$$\text{for } 0 < w \leq 1,\ 0 < q \leq 1,$$

where $q = \exp(-y/\sigma)$ and $w = \exp\{-(x_1^{(1)}-\theta)/\sigma\}$. We maximize it w.r. to θ first, which is equivalent to maximising it w.r. to w. Unlike the Case (II), the **maximum is** not necessarily at a point which is less than $x_1^{(1)}$, but could be at $x_1^{(1)}$ also. Towards this end, consider the function

$$g(w) = (1-qw)^{r_2} w^f, \quad 0 \leq w \leq 1/q,$$

which is plotted in Figure 2.6.1.

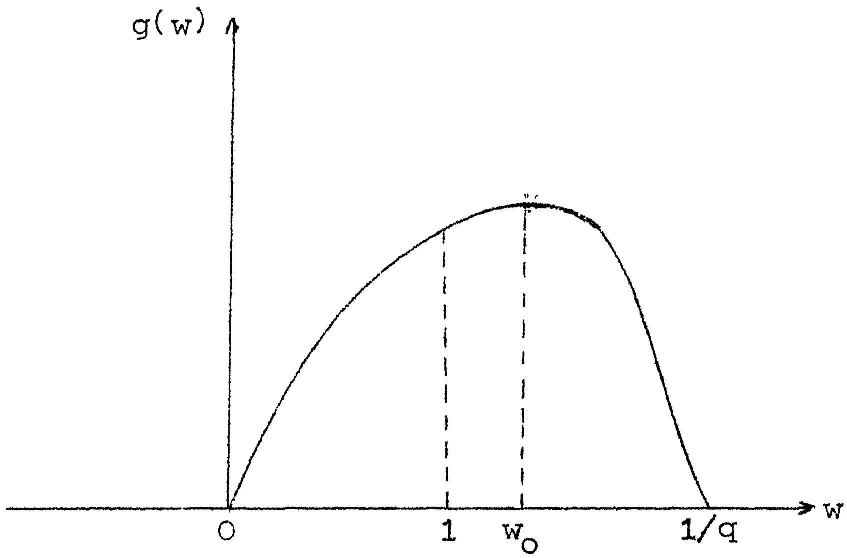

FIGURE 2.6.1. A function related to likelihood function.

Note that $1/q \geq 1$, $g(0) = 0$, $g(1) = (1-q)^{r_2}$ and $g(1/q) = 0$. Differentiating $\log g(w)$ w.r. to w and equating it to zero, we see that the maximum of $g(w)$ in the range 0 to $1/q$ occurs at w_o, where

$$(2.6.14) \qquad w_o = f/\{q(r_2+f)\}.$$

Note that w_o may be greater than 1.

Consequently, the differentiation of likelihood function w.r. to θ gives the ML estimator, only if $w_o \leq 1$. If $w_o > 1$, then the likelihood function will attain the maximum w.r. to θ at a point where $w_o = 1$, that is at $\exp\{-(x_1^{(1)}-\theta)/\sigma\} = 1$. Thus, the maximum in this case is at $\hat{\theta}_o = x_1^{(1)}$. We thus have two possibilities.

<u>Case (i) : $w_o \leq 1$.</u> With same notations as used in equation (2.6.9), we have

$$L(\theta,\sigma \mid \underset{\sim}{z}(1), \underset{\sim}{z}(2))$$
$$= \text{Const.}\ \sigma^{-d^*} [1-e^{-y/\sigma - z_1^{(1)}}]^{r_2} \exp[-\sum_{i=1}^{2}\{s_i z_{n_i-s_i}^{(i)} + \sum_{j=r_i+1}^{n_i-s_i} z_j^{(i)}\}]$$

$$\text{for } z_1^{(1)} \geq 0,\ \sigma > 0.$$

Differentiating $\log L(\theta,\sigma \mid \underset{\sim}{z}(1), \underset{\sim}{z}(2))$ w.r. to θ and σ, we will get ML equations as

$$(2.6.15) \quad \begin{cases} r_2/\{\exp(z_{r_2+1}^{(2)})-1\} - f = 0, \\ r_2 z_{r_2+1}^{(2)}/\{\exp(z_{r_2+1}^{(2)})-1\} - f\, z_1^{(1)} - P/\sigma + d^* = 0. \end{cases}$$

Solving these equations we obtain

(2.6.16) $\quad \hat{\theta}_o = X^{(2)}_{r_2+1} - \hat{\sigma}_o \log \left(\frac{f+r_2}{f}\right), \quad \hat{\sigma}_o = (P-fY)/d^*.$

Case (ii) : $w_o \geq 1$. In this case, as shown above $\hat{\theta}_o = X^{(1)}_1$.

Substituting this in equation (2.6.13), we have to maximize

(2.6.17) $\quad L(\sigma, \hat{\theta}_o | \underset{\sim}{x}(1), \underset{\sim}{x}(2)) = \text{Const.}\ \sigma^{-d^*} \{1-e^{-Y/\sigma}\}^{r_2}\ e^{-P/\sigma}$

(for $y > 0$, $\sigma > 0$.)

w.r. to σ. For this, the maximizing equation is

(2.6.18) $\quad d^*\sigma + r_2 y/\{e^{y/\sigma}-1\} - P = 0.$

The solution of this equation gives the ML estimator $\hat{\sigma}_o$ of σ. This can be solved by using Newton-Raphson method with P/d^* as an initial value.

In Case (III), the procedure of choosing proper ML estimator is as follows :

Calculate the quantities $a = y/\log(1+r_2/f)$ and $b = (P-fy)/d^*$. Depending on the values of 'a' and 'b' we have three possibilities.

Case (1) : $a < b$. In this case, the relevant ML estimators are given in Case (i). Since, the likelihood function attains its maximum within the pertinent range, $w_o \leq 1$. Equivalently

$$\hat{\sigma}_o > y/\log(1+r_2/f),$$

where $\hat{\sigma}_o = b$.

Case (2) : $a > b$. In this case, the ML estimators are given by $\hat{\theta}_o = X_1^{(1)}$ and $\hat{\sigma}_o$ which is the solution of the equation (2.6.18). Note that, for

$$f(\sigma) = d^*\sigma + r_2 y/(e^{y/\sigma}-1) - P, \text{ we have}$$

$$f(0) = -P < 0,$$

$$f(a) = d^*(a-b) > 0$$

and $$f'(\sigma) = d^* + r_2 y^2 e^{y/\sigma}/\{\sigma^2(e^{y/\sigma}-1)^2\} > 0,$$

hence equation (2.6.18) has a unique solution in $(0, a)$.

Case (3) : $a = b$. In this boundary case, the ML estimators given in Case (i) and Case (ii) turn out to be same with $\hat{\theta}_o = X_1^{(1)}$, $\hat{\sigma}_o = a = b$.

Case (IV) : $r_1 > 0$, $r_2 > 0$. Using the same notations as in expression (2.6.9), the likelihood function (2.6.8) can be rewritten as

$$(2.6.19) \quad L(\theta,\sigma \mid \underset{\sim}{z}(1), \underset{\sim}{z}(2)) = \text{Const.}\ \sigma^{-d^*}[1-\exp\{-z_{r_1+1}^{(1)}\}]^{r_1}$$

$$\cdot\ [1-\exp\{-\tfrac{y}{\sigma} - z_{r_1+1}^{(1)}\}]^{r_2} \exp\{-\tfrac{P}{\sigma} - f\, z_{r_1+1}^{(1)}\}$$

$$\text{for } z_{r_1+1}^{(1)} \geq 0,\ \sigma > 0.$$

Consequently,

$$\log L(\theta,\sigma \mid \underset{\sim}{z}(1), \underset{\sim}{z}(2)) = \text{const.} - d^* \log \sigma + r_1 \log [1-\exp\{-z_{r_1+1}^{(1)}\}]$$

$$+ r_2 \log [1-\exp\{-y/\sigma - z_{r_1+1}^{(1)}\}] - P/\sigma - f z_{r_1+1}^{(1)}.$$

The corresponding ML equations are given by

$$\frac{-r_1 \exp\{-z^{(1)}_{r_1+1}\}}{\sigma[1-\exp\{-z^{(1)}_{r_1+1}\}]} - \frac{r_2 \exp\{-z^{(2)}_{r_2+1}\}}{\sigma\,[1-\exp\{-z^{(2)}_{r_2+1}\}]} + \frac{f}{\sigma} = 0$$

and

$$-\frac{d^*}{\sigma} - \frac{r_1 z^{(1)}_{r_1+1} \exp\{-z^{(1)}_{r_1+1}\}}{\sigma\,[1-\exp\{-z^{(1)}_{r_1+1}\}]} - \frac{r_2 z^{(2)}_{r_2+1} \exp\{-z^{(2)}_{r_2+1}\}}{\sigma\,[1-\exp\{-z^{(2)}_{r_2+1}\}]} + \frac{P}{\sigma^2} + \frac{f z^{(1)}_{r_1+1}}{\sigma} = 0.$$

On simplification, these equations reduce to

(2.6.20) $\quad r_1/[\exp\{z^{(1)}_{r_1+1}\}-1] + r_2/[\exp\{z^{(2)}_{r_2+1}\}-1] - f = 0$

and

(2.6.21) $$\frac{r_1 z^{(1)}_{r_1+1}}{\exp\{z^{(1)}_{r_1+1}\}-1} + \frac{r_2 z^{(2)}_{r_2+1}}{\exp\{z^{(2)}_{r_2+1}\}-1} - f z^{(1)}_{r_1+1} - \frac{P}{\sigma} + d^* = 0.$$

Eliminating $[\exp\{z^{(1)}_{r_1+1}\}-1]$ in (2.6.21) by using (2.6.20), we get

(2.6.22) $\quad \theta = x^{(2)}_{r_2+1} - \sigma \log\{1+r_2 y/(P-d^*\sigma)\}.$

Similarly, eliminating $[\exp\{z^{(2)}_{r_2+1}\}-1]$ we obtain

(2.6.23) $\quad \theta = x^{(1)}_{r_1+1} - \sigma \log\{1+r_1 y/(d^*\sigma + fy-P)\}.$

Equating (2.6.22) and (2.6.23), we have

(2.6.24) $$e^{y/\sigma} = [1 + \frac{r_2 y}{P-d^*\sigma}]/[1 + \frac{r_1 y}{d^*\sigma+fy-P}].$$

This equation can be rewritten in the following alternative forms :

(2.6.25) $\sigma = P/d^* - r_2 y / [d^*\{1+r_1 y/(d^*\sigma+fy-P)\} e^{y/\sigma} - d^*]$

or

(2.6.26) $\sigma = (P-fy)/d^* + r_1 y/[d^*\{1+r_2 y/(P-d^*\sigma)\}\ e^{-y/\sigma} - d^*]$.

Remark 2.6.1. Equation (2.6.24) has unique solution for σ, and the solution lies in the interval $((P-fy)/d^*, P/d^*)$. To this end, let

$$f(\sigma) = e^{y/\sigma} - [1+r_2 y/(P-d^*\sigma)]/[1+r_1 y/(d^*\sigma+fy-P)] .$$

Note that,

$$f(P/d^*) = -\infty \text{ and } f((P-fy)/d^*) = \exp\{yd^*/(P-fy)\} > 0.$$

Hence, the equation $f(\sigma) = 0$ has atleast one root in this interval. Now, for concluding it has unique solution, it is sufficient to show that $f(\sigma)$ is monotonically decreasing. Differentiation of $f(\sigma)$ w.r. to σ gives

$$f'(\sigma) = -\exp(y/\sigma) y/\sigma^2$$

$$- \frac{[1+\frac{r_1 y}{(d^*\sigma+fy-P)}\ \ \frac{r_2 yd^*}{(P-d^*\sigma)^2}] - [1+\frac{r_2 y}{(P-d^*\sigma)}][\frac{-r_1 yd^*}{(d^*\sigma+fy-P)^2}]}{[1+r_1 y/(d^*\sigma+fy-P)]^2}$$

< 0 for every $\sigma \in ((P-fy)/d^*, P/d^*)$.

Hence, $f(\sigma)$ is a monotonically decreasing function of σ in the above interval.

Now, the ML estimator of θ is given by one of the equations (2.6.22) or (2.6.23), and from expression (2.6.23),

it is easy to see that $\hat{\theta}_o \leq X^{(1)}_{r_1+1}$, since equation (2.6.26) gives $\hat{\sigma}_o > (P-fy)/d^*$. The ML estimator of σ is the solution of the equation (2.6.24). It is difficult to show by second derivative, that these values really give the maximum of the likelihood function. But some numerical calculations, show that the likelihood function is really maximum at $\hat{\theta}_o$ and $\hat{\sigma}_o$.

<u>Remark 2.6.2.</u> Eventhough $r_1 > 0$, $r_2 > 0$ in Case (IV), we can obtain some results for other cases from this case by substituting r_1 and/or r_2 equal to zero.

(i) If $r_1=r_2=0$ then equation (2.6.25) gives $\hat{\sigma}_o = P/d^*$, as in Case (I).

(ii) If $r_1>0, r_2=0$ then equation (2.6.25) gives $\hat{\sigma}_o = P/d^*$, as in Case (II).

(iii) If $r_1=0$, $r_2>0$ then equation (2.6.26) gives $\hat{\sigma}_o = (P-fy)/d^*$, as in equation (2.6.16).

(iv) If $r_1=0$, $r_2>0$ and $\hat{\theta}_o = X^{(1)}_1$ then equation (2.6.24) reduces to equation (2.6.18).

<u>Remark 2.6.3.</u> A computer program for evaluating the ML estimators for all four cases is given in Appendix A.

2.6.3. <u>Comparison of the estimators</u>.

In general the explicit form of the ML estimators under the hypothesis is complicated. Hence the comparison becomes difficult. However, for $r_1, r_2 = 0$ the ML estimators are given in equation (2.1.12). This can be rewritten as

(2.6.27) $\hat{\theta}_o = \min\{X_1^{(1)}, X_1^{(2)}\}$, $\hat{\sigma}_o = [d\sigma^* + \sum_{i=1}^{2} n_i(X_1^{(i)} - \hat{\theta}_o)]/d^*$.

Before deriving the MSE of the ML estimators, we first prove the following theorem.

Theorem 2.6.1. Let $U = n_1 X_1^{(1)} + n_2 X_1^{(2)} - N\min(X_1^{(1)}, X_1^{(2)})$

and $V = \min(X_1^{(1)}, X_1^{(2)})$,

where $N = n_1 + n_2$, then

(i) $2U/\sigma$ has a χ_2^2 distribution,

(ii) $2N(V-\theta)/\sigma$ has a χ_2^2 distribution,

(iii) The random variables U and V are independently distributed.

Proof. The jpdf of $X_1^{(1)}$ and $X_1^{(2)}$ is given by

$$f(x_1^{(1)}, x_1^{(2)}) = n_1 n_2 \exp\left[-\tfrac{1}{\sigma}\{n_1 x_1^{(1)} + n_2 x_1^{(2)} - N\theta\}\right]/\sigma^2$$

$$\text{for } x_1^{(1)}, x_1^{(2)} > \theta,\ \sigma > 0.$$

Let $u = n_1 x_1^{(1)} + n_2 x_1^{(2)} - N\min\{x_1^{(1)}, x_1^{(2)}\}$ and $v = \min\{x_1^{(1)}, x_1^{(2)}\}$.

This is not a one to one transformation. Two inverse transformations are

1) $x_1^{(1)} = v$, $x_1^{(2)} = u/n_2 + v$

and

2) $x_1^{(2)} = v$, $x_1^{(1)} = u/n_1 + v$,

where $u > 0$ and $v > \theta$. The respective Jacobians of transformations are $1/n_2$ and $1/n_1$. The jpdf of U and V is then

$$f(u,v) = \frac{n_1}{\sigma^2} \exp\left[-\frac{1}{\sigma}\{n_1 v+u+n_2 v-N\theta\}\right]$$

$$+ \frac{n_2}{\sigma^2} \exp\left[-\frac{1}{\sigma}\{n_2 v+u+n_1 v-N\theta\}\right] \text{ for } u > 0,\ v > \theta$$

(2.6.28) $$= \{\exp(-u/\sigma)/\sigma\}\ [N \exp\{-N(v-\theta)/\sigma\}/\sigma],\ u > 0,\ v > \theta.$$

Then the marginal pdf of U and V are

(2.6.29) $$f(u) = \frac{1}{\sigma} e^{-u/\sigma} \text{ for } u > 0$$

and

(2.6.30) $$f(v) = \frac{N}{\sigma} e^{-\frac{N(v-\theta)}{\sigma}} \text{ for } v > \theta$$

respectively. The equation (2.6.29) shows that $2U/\sigma$ has a χ_2^2 distribution. Epstein and Tsao (1953) also obtained the equation (2.6.29) by using a direct argument. Similarly, $2N(V-\theta)/\sigma$ has a χ_2^2 distribution. This can be derived by direct argument as well. From equations (2.6.28), (2.6.29) and (2.6.30), it follows that U and V are independently distributed. This completes the proof of the theorem.

<u>Corollary 2.6.1.</u> With $\hat{\sigma}_o$ and $\hat{\theta}_o$ as defined in equation (2.6.27),

(i) $2N(\hat{\theta}_o-\theta)/\sigma$ has a χ_2^2 distribution,

and

(ii) $2d^*\hat{\sigma}_o/\sigma$ has a $\chi^2_{2(d^*-1)}$ distribution.

<u>Proof.</u> The ML estimators given in equation (2.6.27) can be written as

(2.6.31) $$\hat{\theta}_o = V,\ \hat{\sigma}_o = [2d\sigma^*/\sigma + 2U/\sigma]\ \sigma/2d^*,$$

where U and V are as defined in Theorem 2.6.1. Now, part (i)

of the corollary follows from Theorem 2.6.1. By Corollary 2.3.1, $2d\sigma^*/\sigma$ has a χ^2_{2d} distribution. Since U is a function of $X_1^{(1)}$ and $X_1^{(2)}$, on applying a result similar to Theorem 2.3.1, we see that σ^* and U are independently distributed. Consequently, $2d^*\hat{\sigma}_o/\sigma$ has chi-square distribution with $2d+2 = 2(d^*-1)$DF and the corollary follows.

This corollary gives the following results :

$$E\left[2N(\hat{\theta}_o-\theta)/\sigma\right] = 2 \text{ implying } E(\hat{\theta}_o) = \theta+\sigma/N,$$

$$\text{Var}\left[2N(\hat{\theta}_o-\theta)/\sigma\right] = 4 \text{ implying } \text{Var}(\hat{\theta}_o) = \sigma^2/N^2,$$

$$E\left[2d^*\hat{\sigma}_o/\sigma\right] = 2d^*-2 \text{ implying } E(\hat{\sigma}_o) = (d^*-1)\sigma/d^*$$

and $\text{Var}\left[2d^*\hat{\sigma}_o/\sigma\right] = 4(d^*-1)$ implying $\text{Var}(\hat{\sigma}_o) = (d^*-1)\sigma^2/d^{*2}$.

By these relations we have

$$\text{MSE}(\hat{\theta}_o) = 2\sigma^2/N^2 \tag{2.6.32}$$

and

$$\text{MSE}(\hat{\sigma}_o) = \sigma^2/d^{*2}. \tag{2.6.33}$$

In equation (2.6.7) for $r_1 = r_2 = 0$, the variance of the LS estimators are

$$\text{Var}(\theta_o^*) = d^*\sigma^2/\{d^*(n_1^2+n_2^2)-N^2\} \tag{2.6.34}$$

and

$$\text{Var}(\sigma_o^*) = \sigma^2/\{d^*-N^2/(n_1^2+n_2^2)\}. \tag{2.6.35}$$

Eventhough $\hat{\sigma}_o$ is a biased estimator of σ, from equations (2.6.33) and (2.6.35) it follows that $\text{MSE}(\hat{\sigma}_o)<\text{MSE}(\sigma_o^*)$, where σ_o^* is the unbiased estimator of σ given by equation (2.6.6).

Further, as expected $MSE(\sigma_o^*) \leq MSE(\sigma^*)$ where σ^* is given by equation (2.2.3). This can be seen from the equations (2.2.6) and (2.6.35) and the fact that $N^2/(n_1^2+n_2^2) \leq 2$.

Direct comparison of $MSE(\theta_o^*)$ and $MSE(\hat{\theta}_o)$ seems to be difficult, but for $n_1 = n_2 = n$, the equations (2.6.32) and (2.6.34) simplify to

$$MSE(\hat{\theta}_o) = \sigma^2/2n^2 < MSE(\theta_o^*) = d^*\sigma^2/[(d^*-2)2n^2].$$

Also for $n_1 = 5$, $n_2 = 15$, $s_1 = 0$ and $s_2 = 0$ the equations (2.6.32) and (2.6.34) give

$$MSE(\hat{\theta}_o) = 0.005\ \sigma^2 > MSE(\theta_o^*) = 0.00435\ \sigma^2.$$

This shows that, nothing can be said regarding the preference of the one estimator over the other among θ_o^* and $\hat{\theta}_o$. However, one can choose the estimator with the smaller MSE.

TABLE 2.5.1. The relative efficiency "E" of $\ddot{\sigma}$ w.r. to σ^*

d \ K	1	2	3	4	5	6	8	10
2	1.5000	1.3333	1.1364	1.0000	0.9074	0.8421	0.7576	0.7059
4	1.2500	1.1250	0.9423	0.8300	0.6983	0.6250	0.5294	0.4712
6	1.1667	1.0667	0.9000	0.7576	0.6505	0.5714	0.4667	0.4025
8	1.1250	1.0417	0.8897	0.7570	0.6402	0.5568	0.4444	0.3750
10	1.1000	1.0286	0.8895	0.7538	0.6428	0.5565	0.4378	0.3636
15	1.0667	1.0140	0.9000	0.7763	0.6667	0.5765	0.4464	0.3623
20	1.0500	1.0083	0.9121	0.8000	0.6944	0.6036	0.4667	0.3750
25	1.0400	1.0055	0.9224	0.8205	0.7200	0.6302	0.4894	0.3920
30	1.0333	1.0039	0.9308	0.8377	0.7424	0.6545	0.5121	0.4103

CHAPTER III

TESTING OF HYPOTHESIS ABOUT LOCATION PARAMETERS AGAINST ONE-SIDED ALTERNATIVES

3.1. Introduction and test statistics.

In this chapter a test statistic from type II censored samplesis proposed to test the equality of location parameters of two exponential distributions against one-sided alternatives. The common scale parameter is assumed to be unknown. The null and the non-null distributions of the proposed test statistic are obtained. Some critical points and some values of power are tabulated. An approximation in terms of Student's t distribution for the null case is studied.

Let $X^{(i)}_{r_i+1}, X^{(i)}_{r_i+2}, \ldots, X^{(i)}_{n_i-s_i}$ be two independent samples from $E(\theta_i, \sigma)$ $(i = 1, 2)$. Consider the problem of testing $H_0 : \theta_1 = \theta_2$ against one-sided alternative hypothesis $H_1 : \theta_1 > \theta_2$ or $H_1' : \theta_1 < \theta_2$, when σ is unknown.

For the right censored samples (case $r_1 = r_2 = 0$), Kumar and Patel (1971) have proposed a test statistic (KP test) for testing H_0 against $H_2 : \theta_1 \neq \theta_2$. Weinman *et al.* (1973) extended it for testing H_0 against one-sided alternatives. Against H_1, their test statistic is,

$$W_0 = (X^{(1)}_1 - X^{(2)}_1)/\sigma^* ,$$

where σ^* is the pooled estimator of σ, given in equation (2.2.3). They had obtained the null and non-null distribution of W_o as

$$(3.1.1)\qquad P\left[W_o \le c|H_o\right] = \begin{cases} n_1(1-n_2c/d)^{-d}/n & \text{for } c \le 0 \\ 1-n_2(1+n_1c/d)^{-d}/n & \text{for } c > 0 \end{cases}$$

and

$$(3.1.2)\qquad P\left[W_o \le c|\varphi\right] = \begin{cases} n_1 \exp(-n_2\varphi)(1-n_2c/d)^{-d}/n, & c < 0 \\ Q_d(c_1|0)-n_2e^{n_1\varphi}\, Q_d\{c_1|n_1c/d\}/n & \\ + n_1 \exp(-n_2\varphi)L_d\{c_1|n_2c/d\}/n, & c \ge 0 \end{cases}$$

respectively, where $n = n_1+n_2$, $d = n_1+n_2-r_1-r_2-s_1-s_2-2$,

$$c_1 = d\varphi/c,\ \varphi = (\theta_1-\theta_2)/\sigma,$$

$$Q_p(x|s) = \int_x^\infty y^{p-1}e^{-y-sy}\,dy/(p-1)!, \quad x > 0,\ p = 1,2,\ldots; s > -1$$

and $L_p(x|s) = \int_0^x y^{p-1}e^{-y+sy}dy/(p-1)!$, $x > 0$, $p = 1,2,\ldots$.

The expressions for the critical points obtained by Weinman et al. (given in Section 1.4) are actually interchanged, although his tabulated values are correct. By using the notations as in Section 1.4, the correct critical points of W are

$$w_\alpha^* = \begin{cases} \dfrac{d}{n_1}\left[1 - \left\{\dfrac{n_2}{(n_1+n_2)(1-\alpha)}\right\}^{1/d}\right] & \text{if } n_1(1-\alpha) \le n_2\alpha \\ \dfrac{d}{n_2}\left[\left\{\dfrac{n_1}{(n_1+n_2)\alpha}\right\}^{1/d} - 1\right] & \text{if } n_1(1-\alpha) \ge n_2\alpha. \end{cases}$$

All the foregoing statistics are based on intuitive grounds. On the basis of LS and ML estimators, we propose the following test statistics for testing H_o against H_1 :

$$T_{LSE} = (\theta_1^* - \theta_2^*)/\text{Estimate of } SE(\theta_1^* - \theta_2^*)$$

and

$$T_{MLE} = (\hat{\theta}_1 - \hat{\theta}_2)/\text{Estimate of } SE(\hat{\theta}_1 - \hat{\theta}_2),$$

where θ_i^* and $\hat{\theta}_i$ denote the LS estimator and the ML estimator of θ_i $(i = 1, 2)$ respectively. Using expressions for θ_i^* and $\hat{\theta}_i$ given in equations (2.2.3) and (2.4.3), and expressions for the variances of $(\theta_1^* - \theta_2^*)$ and $(\hat{\theta}_1 - \hat{\theta}_2)$ given in equations (2.2.8) and (2.5.6), these statistics reduce to,

$$(3.1.3) \qquad T_{LSE} = [\{X^{(1)}_{r_1+1} - X^{(2)}_{r_2+1}\}/\sigma^* - q_1]/[V + q_1^2/d]^{\frac{1}{2}}$$

and

$$(3.1.4) \qquad T_{MLE} = [\{X^{(1)}_{r_1+1} - X^{(2)}_{r_2+1}\}/\sigma^* - q_2]/[dV/d^* + q_2^2 d^2/d^{*3}]^{\frac{1}{2}} ,$$

where σ^* is the LS estimator of σ given in equation (2.2.3), $q_1 = b_1 - b_2$, $q_2 = m_1 - m_2$, $V = a_1 + a_2$, $d^* = d+2$,

$$b_i = \sum_{j=1}^{r_i+1} (n_i - j + 1)^{-1}, a_i = \sum_{j=1}^{r_i+1} (n_i - j + 1)^{-2} \text{ and}$$

$$m_i = \log\{n_i/(n_i - r_i)\} \quad \text{for } i = 1, 2.$$

Both of these statistics against one-sided alternative H_1 are equivalent to the statistic

$$(3.1.5) \qquad T = \{X^{(1)}_{r_1+1} - X^{(2)}_{r_2+1}\}/\sigma^* .$$

Against H_1, very large values of T lead to the rejection of H_o, that is, we reject H_o against H_1 if $T \geq c_\alpha$, where c_α is determined so that $P[T \geq c_\alpha | H_o] = \alpha$, and α is the chosen level of significance of the test. Similarly, against H_1', very small values of T lead to the rejection of H_o. We thus need the null distribution of T for finding the critical point c_α, which is derived in Section 3.2. The non-null distribution is obtained in Section 3.3. In Section 3.4, the exact and approximated critical points are evaluated. Power function and its approximation are studied in Section 3.5.

3.2. Null distribution of the statistic T.

We first prove the following lemma :

Lemma 3.2.1. Suppose Z_1 and Z_2 are two independent random variates with pdf of Z_i as

$$(3.2.1) \quad f_{Z_i}(z) = \{1-\exp(-z)\}^{r_i} \exp\{-(n_i-r_i)z\}/B(r_i+1, n_i-r_i), z \geq 0 .$$

Then the pdf of $Z = Z_1 - Z_2$ is given by

$$(3.2.2) \quad g(z) = \begin{cases} H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j) e^{(n_2-r_2+j)z}, & z \leq 0 \\ H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j) e^{-(n_1-r_1+j)z}, & z > 0, \end{cases}$$

where $H = \prod_{i=1}^{2} \{1/B(r_i+1, n_i-r_i)\}$, $f = n_1+n_2-r_1-r_2$ and

$$B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}\, du, \; p > 0, \; q > 0.$$

Proof. The jpdf of Z_1 and Z_2 is

$$f(z_1,z_2) = H(1-e^{-z_1})^{r_1}(1-e^{-z_2})^{r_2} \exp\{-(n_1-r_1)z_1-(n_2-r_2)z_2\}$$

$$\text{for } 0 < z_1, z_2 < \infty.$$

Making a transformation $z = z_1-z_2$ and $z_2 = z_2$, we see that the range of Z_2 and Z are $z_2 > \max(0,-z)$ and $-\infty < z < \infty$ respectively. The inverse transformation is $z_1 = z+z_2$ and $z_2 = z_2$. The jacobian of the transformation is 1, and the jpdf of Z and Z_2 is

$$(3.2.3) \qquad g(z,z_2) = H\left[1-\exp\{-(z_2+z)\}\right]^{r_1} \{1-\exp(-z_2)\}^{r_2}$$

$$\cdot \exp\left[-fz_2-(n_1-r_1)z\right], z_2 > \max(0,-z), -\infty < z < \infty.$$

The marginal pdf of Z is thus

$$(3.2.4) \qquad g(z) = \begin{cases} \int_{-z}^{\infty} g(z,z_2)dz_2 & \text{for } z \leq 0 \\ \\ \int_{0}^{\infty} g(z,z_2)dz_2 & \text{for } z > 0. \end{cases}$$

For $z \leq 0$, on putting $y = z+z_2$, $g(z)$ becomes

$$g(z) = He^{-(n_1-r_1)z} \int_{-z}^{\infty} e^{-fz_2} [1-e^{-(z_2+z)}]^{r_1} (1-e^{-z_2})^{r_2} dz_2$$

$$= He^{-(n_1-r_1)z} \int_{0}^{\infty} e^{-f(y-z)}(1-e^{-y})^{r_1}(1-e^{-y+z})^{r_2} dy .$$

Now expanding $(1-e^{-y+z})^{r_2}$ by binomial expansion and simplifying, we get

$$g(z) = H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} e^{(n_2-r_2+j)z} \int_{0}^{\infty} e^{-(f+j)y} (1-e^{-y})^{r_1} dy.$$

Substituting $u = e^{-y}$ and simplifying, we obtain equation (3.2.2) for $z \leq 0$. For $z > 0$, we expand $[1-e^{-(z_2+z)}]^{r_1}$ of equation (3.2.3) as a binomial sum and get the corresponding result of equation (3.2.2).

Corollary 3.2.1. If $Y = Z + \varphi$, then the **pdf** of Y is

$$(3.2.5)\quad g(y) = \begin{cases} H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j) e^{(n_2-r_2+j)(y-\varphi)} & \text{for } y \leq \varphi \\ H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j) e^{-(n_1-r_1+j)(y-\varphi)} & \text{for } y > \varphi . \end{cases}$$

Proof. The corollary follows from Lemma 3.2.1.

Theorem 3.2.1. The null distribution of T [where T is defined by equation (3.1.5)] is

$$P\,[T \leq c \mid H_o]$$

$$(3.2.6) = \begin{cases} H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j)(1-h_2(j)c)^{-d}/dh_2(j) & \text{for } c \leq 0 \\ 1-H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j)(1+h_1(j)c)^{-d}/dh_1(j) & \text{for } c > 0, \end{cases}$$

where $h_i(j) = (n_i-r_i+j)/d$ $(i = 1, 2)$, $d = f-s_1-s_2-2$, H and f are as in Lemma 3.2.1.

Proof. Note that $Z_i = (X^{(i)}_{r_i+1}-\theta_i)/\sigma$ $(i = 1, 2)$ has the pdf given in equation (3.2.1). Then under $H_o : \theta_1 = \theta_2$, we have $T = dZ/W$, where $Z = Z_1-Z_2$ and $W = d\sigma^*/\sigma$. By Theorem 2.3.1, Z and W are independent with pdf of Z given in Lemma 3.2.1, and that of W

given in equation (2.3.6). Hence, the jpdf of Z and W is

$$g(z,w) = \begin{cases} H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j) e^{(n_2-r_2+j)z} e^{-w} w^{d-1}/(d-1)!, & w>0, z\leq 0 \\ H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j) e^{-(n_1-r_1+j)z} e^{-w} w^{d-1}/(d-1)!, & w>0, z>0. \end{cases}$$

Make a transformation $t = dz/w$, $w = w$. The inverse transformation is $z = tw/d$, $w = w$, and the jacobian of transformation is

$$\left|\frac{\partial(z,w)}{\partial(t,w)}\right| = w/d.$$

The jpdf of T and W is

$$g(t,w) = \begin{cases} H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j) e^{-\{1-h_2(j)t\}w} w^d/d!, & w > 0, t \leq 0 \\ H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j) e^{-\{1+h_1(j)t\}w} w^d/d!, & w > 0, t > 0. \end{cases}$$

Integrating out w, we get the marginal pdf of T as

$$(3.2.7) \quad f(t) = \begin{cases} H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j)\{1-h_2(j)t\}^{-(d+1)}, & t \leq 0 \\ H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j)\{1+h_1(j)t\}^{-(d+1)}, & t > 0. \end{cases}$$

The desired cdf of T now follows from equation (3.2.7).

Note that, for $r_1 = r_2 = 0$, the equation (3.2.6) reduces to equation (3.1.1) given by Weinman _et al_. (1973).

Corollary 3.2.2. With the notations used in Theorem 3.2.1, we have the identity

$$(3.2.8)\quad H\left[\sum_{j=0}^{r_1}(-1)^j\binom{r_1}{j}\frac{B(r_2+1,f+j)}{dh_1(j)}+\sum_{j=0}^{r_2}(-1)^j\binom{r_2}{j}\frac{B(r_1+1,f+j)}{dh_2(j)}\right]=1.$$

Proof. On using the fact that $f(t)$ given in equation (3.2.7) is a pdf, we get the required result.

This corollary can be rewritten as an interesting cominatorial identity, namely

$$\sum_{j=0}^{r_1}(-1)^j\binom{r_1}{j}\frac{B(r_2+1,f+j)}{n_1-r_1+j}+\sum_{j=0}^{r_2}(-1)^j\binom{r_2}{j}\frac{B(r_1+1,f+j)}{n_2-r_2+j}$$

$$=\prod_{i=1}^{2}B(r_i+1,n_i-r_i),$$

where $f = n_1+n_2-r_1-r_2$.

3.3. Non-null distribution of the statistic T.

We next derive the distribution of T under the alternative hypothesis $H_1 : \theta_1 > \theta_2$. This is needed for studying the power function of the test.

Theorem 3.3.1. The non-null cdf of T for $\varphi = (\theta_1-\theta_2)/\sigma \geq 0$ is given by

$$(3.3.1)\qquad P\,[T \leq c|\varphi] = \begin{cases} F_1(c|\varphi), & c < 0 \\ \\ F_2(c|\varphi), & c \geq 0, \end{cases}$$

where

$$F_1(c|\varphi)=H\sum_{j=0}^{r_2}(-1)^j\binom{r_2}{j}B(r_1+1,f+j)\cdot\exp\{-h_2(j)d\varphi\}\cdot\{1-h_2(j)c\}^{-d}/dh_2(j),$$

$$F_2(c|\varphi) = Q_d(c_1|0)-H\sum_{j=0}^{r_1}(-1)^j\binom{r_1}{j}B(r_2+1,f+j)\cdot\exp\{h_1(j)d\varphi\}\cdot$$

$$\cdot\, Q_d\{c_1|h_1(j)c\}/dh_1(j)+H\sum_{j=0}^{r_2}(-1)^j\binom{r_2}{j}B(r_1+1,f+j)\cdot$$

$$\cdot\,\exp\{-h_2(j)d\varphi\}\cdot L_d\{c_1|h_2(j)c\}/dh_2(j)$$

and the remaining notations are given in equation (3.1.2) and in Lemma 3.2.1.

Proof. Let $Z_i = \{X^{(i)}_{r_i+1}-\theta_i\}/\sigma$ $(i = 1,2)$. Then Z_i follows the distribution given in equation (3.2.1), and $Y = Z_1-Z_2+\varphi$ $= \{X^{(1)}_{r_1+1}-X^{(2)}_{r_2+1}\}/\sigma$ has the pdf given in Corollary 3.2.1. Clearly, T can be written as dY/W. Using similar results as in the null case, we now have the jpdf of Y and W as

$$f(y,w) = \begin{cases} f_1(y,w) & \text{for } w > 0,\ y \le \varphi \\ f_2(y,w) & \text{for } w > 0,\ y > \varphi, \end{cases}$$

where

$$f_1(y,w) = H\sum_{j=0}^{r_2}(-1)^j\binom{r_2}{j}B(r_1+1,f+j)e^{(n_2-r_2+j)(y-\varphi)}e^{-w}w^{d-1}/(d-1)!$$

and

$$f_2(y,w) = H\sum_{j=0}^{r_1}(-1)^j\binom{r_1}{j}B(r_2+1,f+j)e^{-(n_1-r_1+j)(y-\varphi)}e^{-w}w^{d-1}/(d-1)!.$$

Making the transformation $t = dy/w$ and $w = w$, the corresponding inverse transformation is $y = wt/d$, $w = w$. The region $w > 0$, $y \le \varphi$ is transformed to $w > 0$, $wt \le d\varphi$. Similarly, the region $w > 0$, $y > \varphi$ is transformed to $w > 0$, $wt > d\varphi$. Note that, the

boundary $y = \varphi$ is transformed to the hyperbola $wt = d\varphi$. (see, Figure 3.3.1). The jacobian of this transformation is w/d, and the jpdf of T and W is

$$g(t,w) = \begin{cases} g_1(t,w) & \text{for } w > 0,\ wt \leq d\varphi \\ g_2(t,w) & \text{for } w > 0,\ wt > d\varphi\ , \end{cases}$$

where

$$g_1(t,w) = H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j)\ e^{h_2(j)(wt-d\varphi)}\ e^{-w}w^d/d!$$

and

$$g_2(t,w) = H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j) e^{-h_1(j)(wt-d\varphi)}\ e^{-w}w^d/d!\ .$$

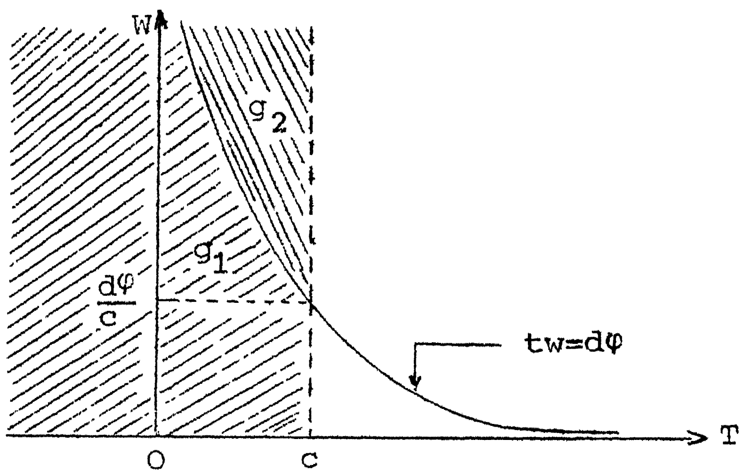

FIGURE 3.3.1. Joint pdf of (T,W) for $\varphi \geq 0$.

The cdf of T upto the point c is the integral of joint density of (T,W) over the shaded region as shown in Figure 3.3.1. This is given by

$$P\left[T \leq c \mid \varphi\right] = \int_0^\infty \left(\int_{-\infty}^{c} g_1 dt\right) dw = I_1 \text{ (say) for } c \leq 0$$

and

$$P\left[T \leq c \mid \varphi\right] = \int_0^\infty \left(\int_{-\infty}^{0} g_1\ dt\right) dw + \int_{d\varphi/c}^{\infty} \left(\int_0^{d\varphi/w} g_1\ dt\right)\ dw$$

$$+ \int_0^{d\varphi/c} \left(\int_0^c g_1 \, dt\right) dw + \int_{d\varphi/c}^{\infty} \left(\int_{d\varphi/w}^{c} g_2 \, dt\right) dw$$

$$= I_2+I_3+I_4+I_5 \text{ (say) for } c > 0.$$

Simplification of I_i's $(i = 1, 2, 3, 4, 5)$ involves lengthy expressions, although the method is straight forward. Simplification of one factor I_4 is illustrated below :

$$I_4 = \int_0^{c_1} \left(\int_0^c g_1 \, dt\right) dw$$

$$= H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j) \frac{e^{-h_2(j)d\varphi}}{dh_2(j)} \int_0^{c_1} \left(e^{h_2(j)wc} - 1\right) \frac{e^{-w} w^{d-1}}{(d-1)!} dw$$

$$= H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j) \frac{e^{-h_2(j)d\varphi}}{dh_2(j)}$$

$$\cdot \left[\int_0^{c_1} \frac{w^{d-1} e^{-w\{1-h_2(j)c\}}}{(d-1)!} dw - \int_0^{c_1} \frac{e^{-w}}{(d-1)!} w^{d-1} dw\right]$$

$$= H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j) \frac{e^{-h_2(j)d\varphi}}{dh_2(j)} \left[L_d(c_1 | h_2(j)c) - 1 + Q_d(c_1 | 0)\right].$$

Similarly,

$$I_1 = H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j) e^{-h_2(j)d\varphi} \{1-h_2(j)c\}^{-d} / dh_2(j),$$

$$I_2 = H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j)\, e^{-h_2(j)d\varphi} / dh_2(j),$$

$$I_3 = H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j) \{1-e^{-h_2(j)d\varphi}\} Q_d(c_1 | 0) / dh_2(j)$$

and

$$I_5 = H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} \frac{B(r_2+1, f+j)}{dh_1(j)} [Q_d(c_1|0)$$

$$- \exp\{h_1(j)d\varphi\} Q_d\{c_1|h_1(j)c\}] .$$

Combining these expressions and using Corollary 3.2.2, the cdf of T as given in equation (3.3.1) is obtained.

For calculation purposes, we need some further simplifications. For this, note that

$$Q_d(x|s) = \sum_{j=0}^{d-1} \exp\{-x(1+s)\}\{x(1+s)\}^j/\{j!(1+s)^d\}, \quad s > -1$$

and

$$L_d(x|s) = \begin{cases} x^d/d! & \text{for } s = 1 \\ [1-\sum_{j=0}^{d-1} \exp\{-x(1-s)\}\{x(1-s)\}^j/j!]/(1-s)^d & \text{for } s \neq 1. \end{cases}$$

The proof of these results and a computer program for evaluating these functions are given in Appendix B.

Substituting these results in expression (3.3.1) we obtain the cdf of T as

$$(3.3.2) \qquad P[T \leq c|\varphi] = P_1(c|\varphi) = \begin{cases} G_1(c|\varphi) & \text{for } c < 0 \\ G_2(c|\varphi) & \text{for } c \geq 0, \end{cases}$$

where

$$G_1(c|\varphi) = H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} \frac{B(r_1+1, f+j) \exp\{-h_2(j)d\varphi\}}{dh_2(j)\{1-h_2(j)c\}^d}$$

and

$$G_2(c|\varphi) = \sum_{i=0}^{d-1} \frac{e^{-c_1}}{i!} c_1^i - H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} \frac{B(r_2+1, f+j)}{dh_1(j)}$$

$$\cdot \sum_{i=0}^{d-1} \frac{c_1^i \exp(-c_1)}{i!\{1+h_1(j)c\}^{d-i}} + H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} \frac{B(r_1+1, f+j)}{dh_2(j)\{1-h_2(j)c\}^d}$$

$$\cdot \left[\exp\{-h_2(j)d\varphi\} - e^{-c_1} \sum_{i=0}^{d-1} \frac{c_1^i}{i!} \{1-h_2(j)c\}^i \right]$$

for $h_2(j)c \neq 1$ $(j = 0,1,2,\ldots,r_2)$. However, for $h_2(j)c = 1$, the $(j+1)$th term of the last factor in $G_2(c|\varphi)$ simply becomes

$$H(-1)^j \binom{r_2}{j} B(r_1+1, f+j) \exp\{-h_2(j)d\varphi\}c_1^d/\{dh_2(j)d!\}.$$

Note that, for $\varphi = 0$, equations (3.3.1) and (3.3.2) reduce to the null distribution of T, given by equation (3.2.6). It is easy to show that, for $r_1 = r_2 = 0$, the equation (3.3.2) reduces to equation (3.1.2), the non-null cdf of the test statistic considered by Weinman et al. (1973).

Usually, the null hypothesis is also taken as one-sided against one-sided alternatives. Thus, if we take $H_0^* : \theta_1 \leq \theta_2$ and test it against $H_1 : \theta_1 > \theta_2$, then we also require the power function for negative values of φ. The following corollary gives the necessary distribution theory results.

Corollary 3.3.1. The non-null cdf of T for $\varphi \leq 0$ is

$$(3.3.3) \qquad P[T \leq c|\varphi] = P_2(c|\varphi) = \begin{cases} M_1(c|\varphi), & c < 0 \\ M_2(c|\varphi), & c \geq 0, \end{cases}$$

where

$$M_1(c|\varphi) = 1-Q_d(c_1|0)+H\sum_{j=0}^{r_2}(-1)^j\binom{r_2}{j}B(r_1+1,f+j)\ \exp\{-h_2(j)d\varphi\}$$

$$\cdot Q_d\{c_1|-h_2(j)c\}/dh_2(j)-H\sum_{j=0}^{r_1}(-1)^j\binom{r_1}{j}B(r_2+1,f+j)$$

$$\cdot\exp\{h_1(j)d\varphi\}\ L_d\{c_1|-h_1(j)c\}/dh_1(j),$$

$$M_2(c|\varphi) = 1-H\sum_{j=0}^{r_1}(-1)^j\binom{r_1}{j}B(r_2+1,f+j)\exp\{h_1(j)d\varphi\}\{1+h_1(j)c\}^{-d}/dh_1(j$$

and the remaining notations are same as in Theorem 3.3.1.

Proof. The proof of the corollary follows on the same lines as that of Theorem 3.3.1, but the cdf of T upto the point c is the integral of joint density of (T,W) over the shaded region as shown in Figure 3.3.2, instead of Figure 3.3.1.

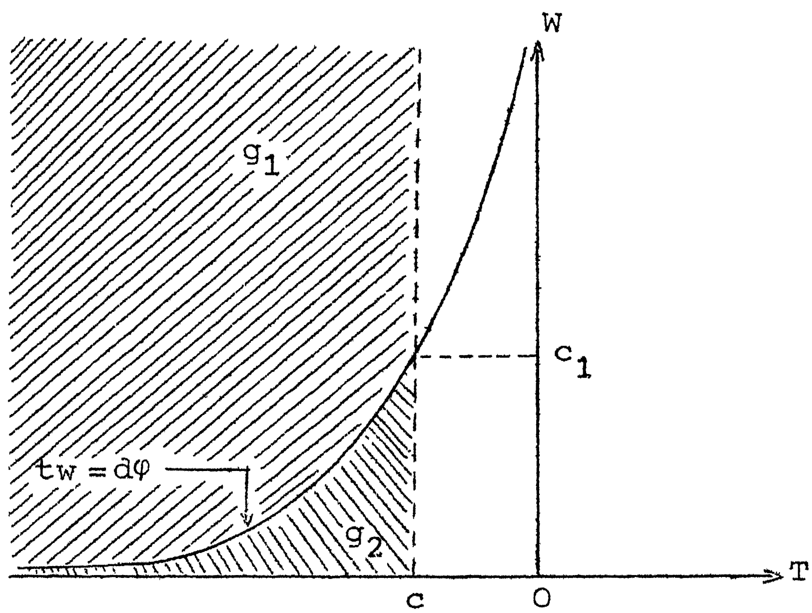

FIGURE 3.3.2. Joint pdf of (T,W) for $\varphi < 0$.

This is given by

$$P\left[T \le c \mid \varphi\right] = \begin{cases} 1-\left[\int_0^\infty \left(\int_0^\infty g_2 dt\right) dw + \int_0^{c_1} \left(\int_c^0 g_2 dt\right) dw \right. \\ \left. + \int_{c_1}^\infty \left(\int_c^0 g_2 dt\right) dw + \int_{c_1}^\infty \left(\int_c^{d\varphi/w} g_1 dt\right) dw\right] & \text{for } c < 0 \\ 1-\int_0^\infty \left(\int_c^\infty g_2 dt\right) dw & \text{for } c \ge 0. \end{cases}$$

On simplification, this gives the equation (3.3.3).

Remark 3.3.1. Although due to the complicated expressions involved, it is not possible to show that the test based on T is unbiased, yet we feel that this test is unbiased. Limited calculations carried out in Section 3.5 strengthen this feeling.

Remark 3.3.2. Critical points and the power values for testing $H_o : \theta_1 = \theta_2$ against $H_1' : \theta_1 < \theta_2$ can be obtained by relabelling the samples.

3.4. Exact and approximate critical points of T.

3.4.1. Exact critical points. The critical point c_α is obtained by solving the equation

$$(3.4.1) \qquad P\left[T \ge c_\alpha \mid H_o\right] = \alpha,$$

where α is the chosen level of significance of the test. From equation (3.2.6) it is clear that $c_\alpha (\le 0)$ is the solution of

$$1-H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j)\{1-h_2(j)c_\alpha\}^{-d}/dh_2(j) = \alpha,$$

if $P_o \ge 1-\alpha$, where

$$P_o = P[T \le 0 | H_o] = H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j)/dh_2(j).$$

Similarly, if $P_o \le 1-\alpha$, then $c_\alpha (\ge 0)$ is the solution of

$$H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j)\{1+h_1(j)c_\alpha\}^{-d}/dh_1(j) = \alpha.$$

Note that, if $r_1 = 0$, then

$$P_o = 1-B(r_2+1, f)/B(r_2+1, n_2-r_2).$$

Now, if $P_o \le 1-\alpha$, then c_α is given by

$$c_\alpha = d\,[\{(1-P_o)/\alpha\}^{1/d} - 1]/n_1 \ge 0.$$

Similarly, if $r_2 = 0$ and

$$P_o = B(r_1+1, f)/B(r_1+1, n_1-r_1) \ge 1-\alpha,$$

then c_α is given by

$$c_\alpha = d\,[1-\{P_o/(1-\alpha)\}^{1/d}]/n_2 \le 0.$$

In particular, if $r_1 = r_2 = 0$, then $P_o = n_1/(n_1+n_2)$ and

$$c_\alpha = \begin{cases} \dfrac{d}{n_2}\left[1 - \left\{\dfrac{n_1}{(n_1+n_2)(1-\alpha)}\right\}^{1/d}\right] & \text{if } n_1\alpha \ge n_2(1-\alpha) \\[2ex] \dfrac{d}{n_1}\left[\left\{\dfrac{n_2}{(n_1+n_2)\alpha}\right\}^{1/d} - 1\right] & \text{if } n_1\alpha \le n_2(1-\alpha). \end{cases}$$

Other than these cases, c_α can be obtained by evaluating P_o and then applying Newton - Raphson method to the pertinent equation. Note that if $r_1 < r_2$ then it may be more convenient to calculate $P[T \ge 0 | H_o]$. Some critical points are tabulated

in Table 3.4.1 and Table 3.4.2 for $\alpha = 0.05$. These are tabulated for following combinations of sample sizes and censoring patterns:

(i) for $n_1=n_2=10$, $(r_1,r_2)=(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)$ and $d = 4(2)16$ in Table 3.4.1, and

(ii) for $n_1=10$, $n_2=6(2)24$, $(r_1,r_2)=(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)$ and $d = 12$ in Table 3.4.2.

Note that s_1 and s_2 appear only through d. Due to many variables involved, it is not possible to give a large table of critical points. In these tables, we have provided mostly those values which were later used for studying the power function of the test.

Initial value for solving equation (3.4.1) can be taken as an approximate critical point, given in Section 3.4.2.

3.4.2. <u>Student's t approximation.</u> The asymptotic distribution of sample quantiles $X^{(1)}_{r_1+1}$ and $X^{(2)}_{r_2+1}$ is well known (for example, see David 1981, p. 255). We however use the exact mean and variance of $Z = X^{(1)}_{r_1+1}-X^{(2)}_{r_2+1}$ under H_o. From the results given in Section 2.2, we have

$$(3.4.2) \qquad E(Z) = B\sigma, \ Var(Z) = A\sigma^2,$$

where $B = b_1-b_2$, $A = a_1+a_2$, $a_i = \sum_{j=1}^{r_i+1} (n_i-j+1)^{-2}$,

$$b_i = \sum_{j=1}^{r_i+1} (n_i-j+1)^{-1} \quad (i = 1,2).$$

Using the asymptotic normality we see that

$$[\{X^{(1)}_{r_1+1}-X^{(2)}_{r_2+1}\}/\sigma-B]/\sqrt{A} \overset{d}{=} AN(0,1).$$

Now, following Tiku (1981), we can approximate the null distribution of

(3.4.3) $$[\{X^{(1)}_{r_1+1}-X^{(2)}_{r_2+1}\}/\sigma^*-B]/\sqrt{A} = (T-B)/\sqrt{A}$$

as Student's t distribution with d DF. Therefore, the exact critical value c_α is approximately given by

(3.4.4) $$c_\alpha \cong c^*_\alpha \sqrt{A} + B = c^*_1 ,$$

where c^*_α is the upper αth percentile point of Student's t distribution with d DF.

3.4.3. Normal approximation. We now consider a normal approximation for the null distribution. For this we need the mean and variance of T.

From Corollary 2.3.1, $W = 2d\sigma^*/\sigma$ has a χ^2_{2d}. Hence

(3.4.5) $$E[1/W^m] = (d-m-1)!/[2^m(d-1)!], \quad m = 1, 2, \ldots, (d-1).$$

Now, T can be written as

(3.4.6) $$T = 2dZ/(\sigma W).$$

From the independence of Z and W, and using equation (3.4.2), we get

(3.4.7) $$E(T) = dB/(d-1), \quad Var(T) = d^2\{(d-1)A+B^2\}/\{(d-1)^2(d-2)\}.$$

We now use the fact that

$$\{T-E(T)\}/\{Var(T)\}^{1/2} \overset{d}{=} AN(0,1),$$

where E(T) and Var(T) are given in equation (3.4.7). Hence,

$$(3.4.8) \qquad c_\alpha \cong c_\alpha^{**}\{Var(T)\}^{1/2} + E(T) = c_2^*,$$

where c_α^{**} is the upper αth percentile point of normal distribution.

The exact values c_α and the approximated values c_1^* and c_2^* obtained from the equations (3.4.4) and (3.4.8) are tabulated in Table 3.4.3, for $\alpha = 0.05$ and for some selected values of n_1, n_2, r_1, r_2 and d. On the basis of this and some other calculations, it is observed that, in general normal approximation value c_2^* is better than c_1^* for $r_1 > r_2$, otherwise c_1^* is better than c_2^*.

3.5. Power function and its approximation.

The power of the test T for testing $H_o^* : \theta_1 \leq \theta_2$ against $H_1 : \theta_1 > \theta_2$ is given by

$$(3.5.1) \qquad P\left[T \geq c_\alpha | \varphi\right] = \begin{cases} 1-P_1(c_\alpha | \varphi) & \text{for} \quad \varphi \geq 0 \\ 1-P_2(c_\alpha | \varphi) & \text{for} \quad \varphi < 0, \end{cases}$$

where c_α is the exact critical point, $P_1(c_\alpha | \varphi)$ and $P_2(c_\alpha | \varphi)$ are given in equations (3.3.2) and (3.3.3) respectively. For $\alpha = 0.05$, the power of the test T is tabulated for various values of φ in Table 3.5.1, Table 3.5.2 and Table 3.5.3 for

the following combinations of sample sizes and censoring patterns :

(i) for $n_1=n_2=10, d=16$ and $(r_1, r_2)=(0,0),(0,1),(0,2),(1,0),(1,1),(2,0)$ in Table 3.5.1,

(ii) for $n_1=n_2=10, d=4(2)16$ and $r_1=r_2=1$ in Table 3.5.2, and

(iii) for $n_1=10, n_2=6(2)16$ and $r_1=r_2=1$ in Table 3.5.3.

The variation in power due to different combination of r_1 and r_2 is showed in the Figure 3.5.1. From this and some other calculations the following points emerge.

(a) Table 3.5.1 and Figure 3.5.1 show that the test T is more sensitive for variation in r_1 compared to variation in r_2. Consequently, it is desirable to take greater care in handling first sample so that censoring on the left is reduced to a minimum for this sample.

(b) It is clear from the Table 3.5.2 that for fixed n_1, n_2, r_1 and r_2 the power of the test is relatively less affected by variation in right truncation which is represented by variations in d.

(c) Table 3.5.3 shows that for fixed r_1, r_2 and d, the power of test is not much affected by increasing the sample size.

Since the power function given in equation (3.5.1) is very complicated, for large d and moderate critical points (c_α) the following normal approximation is suggested.

Let $U = X^{(1)}_{r_1+1} - X^{(2)}_{r_2+1} - c_\alpha \sigma^*$. Note that, for large d, a normal approximation of chi square distribution yields

$\sigma^* \overset{d}{=} AN(\sigma, \sigma^2/d)$ and for moderate value of c_α the effect of $c_\alpha\sigma^*$ is not negligible in the linear combination defining U. Then from Theorem 2.3.1 and Corollary 2.3.1, we see that under H_1

$$E(U) = (\varphi+B-c_\alpha)\sigma \text{ and } Var(U) = (A+c_\alpha^2/d)\sigma^2,$$

where $\varphi = (\theta_1-\theta_2)/\sigma$. Therefore,

$$P\left[T \geq c_\alpha|\varphi\right] = P\left[(X^{(1)}_{r_1+1}-X^{(2)}_{r_2+1})/\sigma^* \geq c_\alpha|\varphi\right]$$

$$= P\left[U \geq 0|\varphi\right]$$

$$(3.5.2) \qquad \cong 1 - \Phi(b^*),$$

where $b^* = -(\varphi+B-c_\alpha)/(A+c_\alpha^2/d)^{1/2}$ and $\Phi(x) = \int_{-\infty}^{x} e^{-y^2/2}\, dy/\sqrt{(2\pi)}$.

The exact power [left hand side of equation (3.5.2)] and the approximated power [right hand side of equation (3.5.2)] are tabulated in Table 3.5.4 for $\alpha = 0.05$. As can be seen from this table, the normal approximation is quite satisfactory in this case. However, some calculations performed for small values of c_α show that this approximation is not that good for such situations.

TABLE 3.4.1. Exact upper critical points c_α of the test statistic T for $\alpha = 0.05$ and $n_1=n_2=10$.

$d \backslash (r_1, r_2)$	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(2,0)
4	0.3113	0.1901	0.0818	0.6401	0.4780	1.0100
6	0.2807	0.1776	0.0793	0.5575	0.4280	0.8614
8	0.2668	0.1717	0.0780	0.5209	0.4054	0.7960
10	0.2589	0.1683	0.0773	0.5002	0.3925	0.7593
12	0.2538	0.1661	0.0768	0.4870	0.3842	0.7358
14	0.2503	0.1645	0.0765	0.4778	0.3784	0.7195
16	0.2477	0.1633	0.0762	0.4711	0.3742	0.7075

TABLE 3.4.2. Exact upper critical points c_α of the test statistic T for $\alpha = 0.05$, $n_1=10$ and $d = 12$.

$n_2 \backslash (r_1, r_2)$	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(2,0)
6	0.2194	0.0952	-1.4357	0.4436	0.2928	0.6863
8	0.2396	0.1370	0.0321	0.4693	0.3474	0.7154
10	0.2538	0.1661	0.0768	0.4870	0.3842	0.7358
12	0.2644	0.1876	0.1097	0.5000	0.4110	0.7506
14	0.2726	0.2043	0.1351	0.5099	0.4315	0.7618
16	0.2792	0.2176	0.1553	0.5178	0.4476	0.7707
18	0.2846	0.2285	0.1718	0.5242	0.4606	0.7772
20	0.2891	0.2375	0.1856	0.5295	0.4714	0.7836
22	0.2929	0.2453	0.1972	0.5339	0.4805	0.7885
24	0.2962	0.2519	0.2072	0.5377	0.4882	0.7927

TABLE 3.4.3. Comparison of exact (c_α), Student's t approximation (c_1^*) and normal approximation (c_2^*) critical points of the test T for α =0.05.

r_1	r_2	$n_1=n_2=10$, d = 13			$n_1=n_2=15$, d = 22		
		c_α	c_1^*	c_2^*	c_α	c_1^*	c_2^*
0	0	.2519	.2505	.2632	.1618	.1620	.1665
0	1	.1652	.2074	.2197	.1088	.1318	.1359
0	2	.0766	.1518	.1712	.0548	.0941	.1002
1	0	.4820	.4296	.4604	.2989	.2746	.2855
1	1	.3811	.3744	.3935	.2393	.2374	.2440
2	0	.7270	.6240	.6827	.4358	.3908	.4111

TABLE 3.5.1. Power of the test T for testing H_o against H_1, for α = 0.05, $n_1=n_2=10$ and d = 16.

φ \ (r_1, r_2)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(2,0)
-0.50	.0003	.0003	.0003	.0009	.0009	.0017
-0.40	.0009	.0009	.0009	.0020	.0020	.0035
-0.30	.0025	.0025	.0025	.0046	.0046	.0071
-0.20	.0068	.0068	.0068	.0104	.0105	.0139
-0.10	.0184	.0184	.0184	.0231	.0232	.0267
-0.05	.0303	.0303	.0303	.0341	.0342	.0367
0.00	.0500	.0500	.0500	.0500	.0500	.0500
0.05	.0824	.0824	.0824	.0726	.0725	.0675
0.10	.1358	.1358	.1355	.1042	.1038	.0903
0.15	.2231	.2218	.2146	.1476	.1467	.1194
0.20	.3540	.3416	.3136	.2054	.2035	.1559
0.25	.5137	.4758	.4205	.2792	.2754	.2006
0.30	.6662	.6016	.5249	.3683	.3613	.2539
0.35	.7851	.7070	.6198	.4684	.4566	.3155
0.40	.8665	.7895	.7020	.5721	.5543	.3843
0.45	.9183	.8514	.7703	.6709	.6472	.4581
0.50	.9504	.8966	.8256	.7577	.7294	.5341
0.60	.9817	.9514	.9029	.8829	.8528	.6803
0.70	.9933	.9778	.9479	.9498	.9259	.8016
0.80	.9975	.9901	.9728	.9301	.9646	.8831
0.90	.9991	.9956	.9362	.9924	.9836	.9421
1.00	.9997	.9931	.9931	.9972	.9926	.9722

TABLE 3.5.2. Power of the test T for testing H_0 against H_1, for $\alpha = 0.05$, $n_1=n_2=10$ and $r_1=r_2=1$.

φ \ d	4	6	8	10	12	14	16
−0.50	.0010	.0009	.0009	.0009	.0009	.0009	.0009
−0.40	.0022	.0021	.0021	.0021	.0021	.0020	.0020
−0.30	.0050	.0048	.0048	.0047	.0047	.0047	.0046
−0.20	.0110	.0108	.0107	.0106	.0105	.0105	.0105
−0.10	.0239	.0236	.0234	.0233	.0233	.0232	.0232
−0.05	.0347	.0345	.0344	.0343	.0343	.0342	.0342
0.00	.0500	.0500	.0500	.0500	.0500	.0500	.0500
0.05	.0710	.0716	.0720	.0722	.0723	.0724	.0725
0.10	.0992	.1012	.1022	.1029	.1033	.1037	.1038
0.15	.1357	.1403	.1428	.1444	.1454	.1462	.1467
0.20	.1808	.1901	.1952	.1985	.2007	.2023	.2035
0.25	.2341	.2504	.2597	.2658	.2700	.2732	.2754
0.30	.2942	.3198	.3348	.3449	.3520	.3573	.3613
0.35	.3592	.3955	.4173	.4320	.4425	.4505	.4566
0.40	.4266	.4740	.5026	.5219	.5353	.5463	.5543
0.45	.4942	.5519	.5863	.6093	.6256	.6378	.6472
0.50	.5600	.6260	.6644	.6894	.7069	.7197	.7294
0.60	.6795	.7539	.7937	.8173	.8337	.8448	.8528
0.70	.7772	.8437	.8823	.9016	.9131	.9207	.9259
0.80	.8512	.9123	.9376	.9501	.9572	.9616	.9646
0.90	.9041	.9517	.9685	.9759	.9798	.9821	.9836
1.00	.9401	.9745	.9847	.9887	.9907	.9919	.9926

TABLE 3.5.3. Power of the test T for testing H_o against H_1, for $\alpha = 0.05$, $n_1 = 10$, $d = 12$ and $r_1 = r_2 = 1$.

φ \ n_2	6	8	10	12	14	16
−0.50	.0009	.0009	.0009	.0009	.0009	.0009
−0.40	.0021	.0021	.0021	.0020	.0020	.0020
−0.30	.0048	.0047	.0047	.0047	.0047	.0046
−0.20	.0107	.0106	.0105	.0105	.0105	.0105
−0.10	.0235	.0233	.0233	.0232	.0232	.0232
−0.05	.0344	.0343	.0343	.0342	.0342	.0342
0.00	.0500	.0500	.0500	.0500	.0500	.0500
0.05	.0718	.0721	.0723	.0724	.0724	.0725
0.10	.1018	.1028	.1033	.1036	.1037	.1038
0.15	.1418	.1442	.1454	.1461	.1464	.1466
0.20	.1929	.1981	.2007	.2021	.2028	.2033
0.25	.2550	.2650	.2700	.2727	.2742	.2751
0.30	.3256	.3430	.3520	.3568	.3595	.3612
0.35	.4010	.4283	.4425	.4504	.4548	.4576
0.40	.4771	.5153	.5358	.5472	.5538	.5578
0.45	.5502	.5990	.6256	.6405	.6492	.6544
0.50	.6179	.6754	.7069	.7247	.7349	.7411
0.60	.7323	.7983	.8337	.8535	.8647	.8713
0.70	.8179	.8811	.9131	.9301	.9394	.9447
0.80	.8787	.9325	.9572	.9693	.9756	.9789
0.90	.9205	.9627	.9798	.9874	.9910	.9928
1.00	.9485	.9797	.9907	.9950	.9968	.9977

TABLE 3.5.4. Exact and approximated power of the test T_r for $\alpha = 0.05, n_1=n_2=10$ and $d = 16$.

c_α	0.0762		0.3742		0.7075	
(r_1, r_2)	(0,2)		(1,1)		(2,0)	
φ	Exact	Approx.	Exact	Approx.	Exact	Approx.
−0.50	.0003	.0001	.0009	.0001	.0017	.0003
−0.40	.0009	.0006	.0020	.0004	.0035	.0010
−0.30	.0025	.0027	.0046	.0018	.0071	.0031
−0.20	.0068	.0099	.0105	.0065	.0139	.0085
−0.10	.0184	.0304	.0232	.0201	.0267	.0212
−0.05	.0303	.0497	.0342	.0333	.0367	.0320
0.00	.0500	.0777	.0500	.0528	.0500	.0470
0.05	.0824	.1164	.0725	.0804	.0675	.0672
0.10	.1355	.1671	.1038	.1178	.0903	.0935
0.15	.2146	.2302	.1467	.1661	.1194	.1268
0.20	.3136	.3047	.2035	.2256	.1559	.1675
0.25	.4205	.3884	.2754	.2955	.2006	.2158
0.30	.5249	.4777	.3613	.3741	.2539	.2713
0.35	.6198	.5681	.4566	.4583	.3155	.3332
0.40	.7020	.6550	.5543	.5444	.3843	.3999
0.45	.7703	.7344	.6472	.6285	.4581	.4697
0.50	.8256	.8034	.7294	.7068	.5341	.5405
0.60	.9029	.9047	.8528	.8357	.6803	.6761
0.70	.9479	.9611	.9259	.9206	.8016	.7916
0.80	.9728	.9867	.9646	.9673	.8881	.8784
0.90	.9862	.9962	.9836	.9885	.9421	.9361
1.00	.9931	.9991	.9926	.9966	.9722	.9698

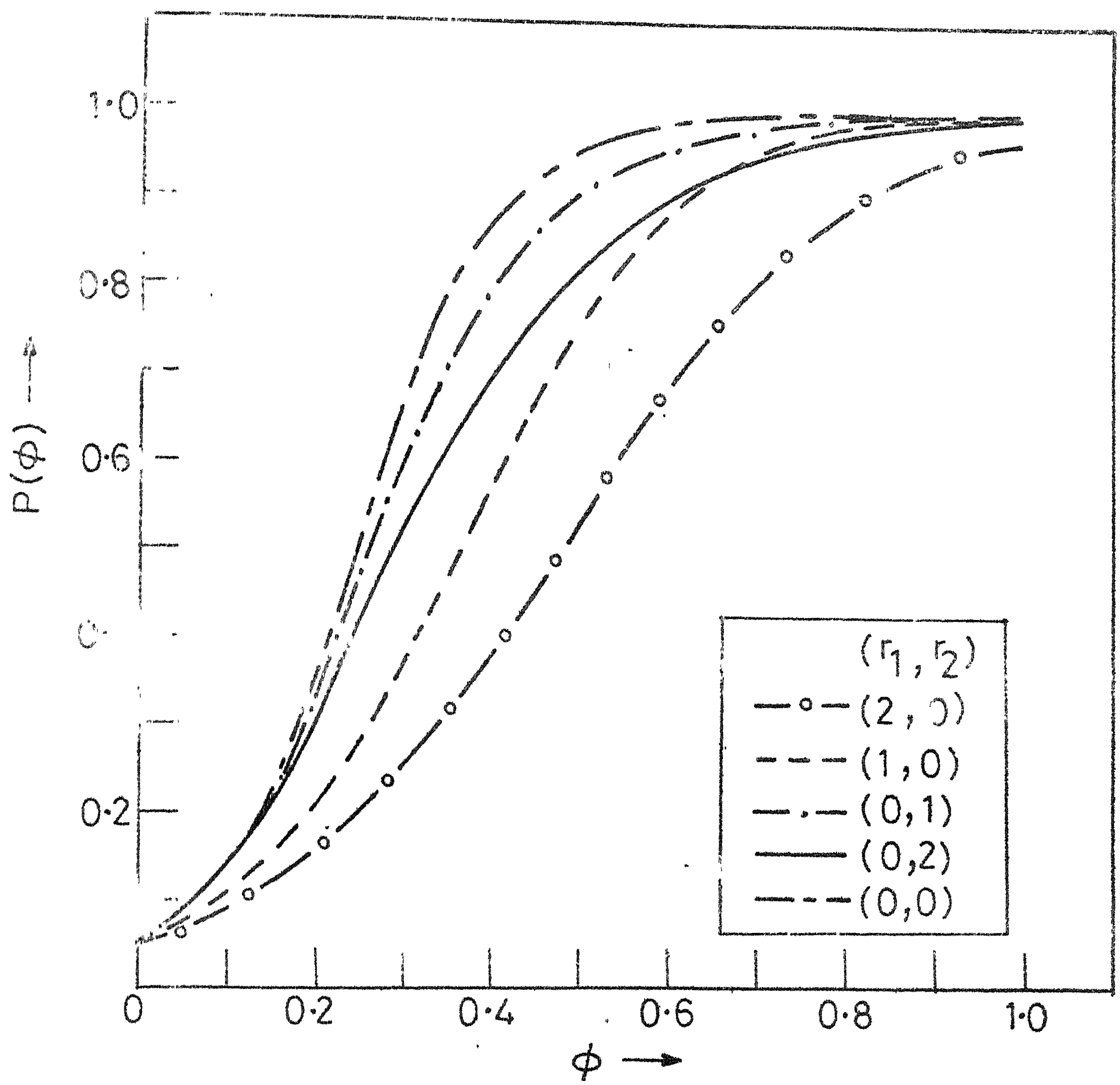

FIGURE 3·5·1. Power functions of T for $\alpha = \cdot 05$ $n_1 = n_2 = 10$ and $d = 16$

CHAPTER IV

TESTS OF HYPOTHESIS FOR LOCATION PARAMETERS AGAINST TWO-SIDED ALTERNATIVE

4.1. Introduction and test statistics.

In this chapter, test statistics based on the LS and ML estimators are proposed for testing $H_o : \theta_1 = \theta_2$, against the two-sided alternative $H_2 : \theta_1 \neq \theta_2$. The null and non-null distributions of the proposed test statistics are derived in Section 4.2. In Section 4.3, the LR test statistic for testing H_o against H_2 is discussed. An approximation to the critical point is investigated in Section 4.4. In Section 4.5, the performance of tests is studied and comparisons with Tiku's (1981) test and LR test are made.

Epstein and Tsao (1953) discussed the LR test based on right censored samples. For same problem, Kumar and Patel (1971), also proposed a test (KP test) based on $U_1 = |(X_1^{(1)} - X_1^{(2)})/\sigma^*|$, where σ^* is the pooled estimator of σ, given by equation (2.2.3). They obtained the null distribution and have tabulated some critical points of U_1. Dubey (1973) and Weinman et al. (1973) derived the power function of the KP test. Weinman et al. also compared the KP test with LR test. They used the power function expression for the LR test as given by Paulson (1941).

Recently, Tiku (1981) generalised the KP test for left censored samples by proposing a test statistic

$$U = |(X_{r_1+1}^{(1)} - X_{r_2+1}^{(2)})/\sigma^*| .$$

He obtained the null distribution of U. Further, he studied a Student's t approximation of the null distribution to obtain approximated critical points. Khatri (1981) derived its non-null distribution. There are some practical limitations for evaluating the power function of the test by using the non-null pdf given by Khatri. As a particular case of our statistic, the power function of Tiku's statistic (U) is given in Section 4.2.

For the one-sided case we have obtained the statistics T_{LSE} and T_{MLE}, given by equations(3.1.3) and (3.1.4) respectively. Both are equivalent to statistic T given in equation (3.1.5). For testing $H_o : \theta_1 = \theta_2$ against $H_2 : \theta_1 \neq \theta_2$ we may use either

$$(4.1.1) \quad V_{LSE} = |(X_{r_1+1}^{(1)} - X_{r_2+1}^{(2)})/\sigma^* - q_1| = |T - q_1| = V_1 \text{ (say)},$$

or

$$(4.1.2) \quad V_{MLE} = |(X_{r_1+1}^{(1)} - X_{r_2+1}^{(2)})/\sigma^* - q_2| = |T - q_2| = V_2 \text{ (say)},$$

where $q_1 = b_1 - b_2$, $q_2 = m_1 - m_2$,

$$b_i = \sum_{j=1}^{r_i+1} (n_i - j + 1)^{-1} \text{ and } m_i = \log\{n_i/(n_i - r_i)\} \quad (i = 1, 2).$$

Note that, $q_2 = 0$ if $r_1/n_1 = r_2/n_2$. We have not been able to identify cases where $q_1 = 0$ except for the trivial case of $r_1 = r_2$ and $n_1 = n_2$. These statistics are of the form

$$(4.1.3) \quad V = |(X_{r_1+1}^{(1)} - X_{r_2+1}^{(2)})/\sigma^* - q| = |T - q| ,$$

where q is a suitable constant which is either equal to q_1 or equal to q_2. Without loss of generality, we can take $q \geq 0$, since

$$V = |(X^{(2)}_{r_2+1} - X^{(1)}_{r_1+1})/\sigma^* - q^*|,$$

where $q^* = -q$, and the distribution is obtained by interchanging the samples.

For $q = 0$, V reduces to the Tiku's statistic U. Further, for $r_1 = r_2 = 0$ and $n_1 = n_2$ the test statistics V_{LSE}, V_{MLE}, U, U_1 and the LR test given by Epstein and Tsao (1953) are equivalent.

The test procedure is to reject H_o against H_2, if $V \geq c_\alpha$, where c_α is determined so that $P[V \geq c_\alpha | H_o] = \alpha$, and α is the chosen level of significance.

4.2. Distribution theory.

This section is devoted to deriving the null and non-null distributions of the statistic V and its special cases.

Theorem 4.2.1. For $q \geq 0$, the cdf of V under H_o is

$$(4.2.1) \quad P[V \leq c | H_o] = \begin{cases} G_1(c) & , \quad 0 \leq c < q \\ G_2(c) & , \quad q \leq c < \infty, \end{cases}$$

where

$$G_1(c) = H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j) \, [\{1+h_1(j)(q-c)\}^{-d}$$

$$- \{1+h_1(j)(q+c)\}^{-d}]/dh_1(j),$$

$$G_2(c) = 1-H\sum_{j=0}^{r_1}(-1)^j\binom{r_1}{j}B(r_2+1,f+j)\{1+h_1(j)(q+c)\}^{-d}/dh_1(j)$$

$$-H\sum_{j=0}^{r_2}(-1)^j\binom{r_2}{j}B(r_1+1,f+j)\{1-h_2(j)(q-c)\}^{-d}/dh_2(j)$$

and the remaining notations are same as in Theorem 3.2.1.

Proof. Note that $V = |T-q|$, where T has pdf given in equation (3.2.6). For the random variable $Y = T-q$, the cdf is given by

$$P[Y \leq y|H_o] = \begin{cases} F_1(y) & \text{for } y \leq -q \\ F_2(y) & \text{for } y > -q, \end{cases}$$

where $F_1(y) = H\sum_{j=0}^{r_2}(-1)^j\binom{r_2}{j}B(r_1+1,f+j)\{1-h_2(j)(q+y)\}^{-d}/dh_2(j)$

and $F_2(y) = 1-H\sum_{j=0}^{r_1}(-1)^j\binom{r_1}{j}B(r_2+1,f+j)\{1+h_1(j)(q+y)\}^{-d}/dh_1(j)$.

Since $V = |Y|$, $P[V \leq c|H_o] = P[Y \leq c|H_o] - P[Y \leq -c|H_o]$.

Consequently, for $q \geq 0$,

$$P[V \leq c|H_o] = \begin{cases} 0 & \text{for } c < 0 \\ F_2(c)-F_2(-c) & \text{for } 0 \leq c \leq q \\ F_2(c)-F_1(-c) & \text{for } q \leq c < \infty. \end{cases}$$

On simplification, equation (4.2.1) follows.

The above theorem immediately gives Corollary 4.2.1 for $q = 0$, which agrees with the null distribution derived by Tiku (1981).

Corollary 4.2.1. For $q = 0$, the cdf of Tiku's test statistic U under H_o is

$$(4.2.2)\quad P\,[U \le c|H_o] = 1-H[\sum_{j=0}^{r_1}(-1)^j\binom{r_1}{j}B(r_2+1,f+j)\{1+h_1(j)c\}^{-d}$$

$$\cdot\ 1/dh_1(j)+\sum_{j=0}^{r_2}(-1)^j\binom{r_2}{j}B(r_1+1,f+j)\{1+h_2(j)c\}^{-d}/dh_2(j)]$$

for $0 \le c < \infty$.

For the KP test, we have $q = 0$ and $r_1 = r_2 = 0$. The cdf is now given in Corollary 4.2.2. This agrees with the cdf obtained by Kumar and Patel (1971).

Corollary 4.2.2. For $q = 0$, $r_1 = r_2 = 0$, we have the cdf of KP test statistic U_1 as

$$(4.2.3)\quad P\,[U_1 \le c|H_o] = 1-\{n_2(1+n_1c/d)^{-d}+n_1(1+n_2c/d)^{-d}\}/(n_1+n_2),$$

for $0 \le c < \infty$.

The distribution of the statistic V under $H_2 : \theta_1 \ne \theta_2$ can be obtained directly from the distribution of T given in Theorem 3.3.1. The derivation of the non-null distribution of V is similar to that of the null distribution of V obtained in Theorem 4.2.1. This distribution is given in the following theorem:

Theorem 4.2.2. For $\varphi = (\theta_1-\theta_2)/\sigma \ge 0$ and $q \ge 0$, the cdf of V under H_2 is given by

$$(4.2.4)\quad P\,[V \le c|\varphi] = P_1(c|\varphi)$$

$$= \begin{cases} F_2(q+c|\varphi)-F_2(q-c|\varphi), & 0 \le c < q \\ F_2(q+c|\varphi)-F_1(q-c|\varphi), & q \le c < \infty, \end{cases}$$

where $F_1(\cdot|\varphi)$ and $F_2(\cdot|\varphi)$ are given in equation (3.3.1).

The equation (4.2.4) reduces to equation (4.2.1), the null cdf of V for $\varphi = 0$. For $q = 0$, Theorem 4.2.2 gives the non-null cdf of the Tiku's statistic.

Corollary 4.2.3. The non-null cdf of the Tiku's statistic is

$$(4.2.5) \quad P[U \leq c|\varphi] = Q_d(c_1|0) - H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j)$$

$$\cdot \exp\{h_1(j)d\varphi\} \; Q_d\{c_1|h_1(j)c\}/dh_1(j) + H \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j}$$

$$\cdot B(r_1+1, f+j) \; \exp\{-h_2(j)d\varphi\} \; [L_d\{c_1|h_2(j)c\}$$

$$-\{1+h_2(j)c\}^{-d}]/dh_2(j) \text{ for } 0 \leq c < \infty,$$

where the notations are same as in Theorem 3.3.1.

The following lemma can be used to obtain the non-null pdf of V as well as U.

Lemma 4.2.1. Let $c_1 = d\varphi/c$, $\varphi \geq 0$ and $Q_d(x|s)$ and $L_d(x|s)$ be as defined in equation (3.1.2). The first derivative of $Q_d(\cdot|\cdot)$ and $L_d(\cdot|\cdot)$ functions w.r. to c are given by

$$(i) \quad Q_d'(c_1|0) = c_1^d \, e^{-c_1}/c\,\Gamma(d),$$

$$(ii) \quad Q_d'\{c_1|h_1 c\} = -dh_1 \sum_{i=0}^{d} \frac{e^{-c_1(1+h_1 c)}\{c_1(1+h_1 c)\}^i}{(1+h_1 c)^{d+1} \; i!}$$

$$+ c_1^d \, e^{-c_1\{1+h_1 c\}}/c\,\Gamma(d)$$

and

(iii) $$L'_d(c_1|h_2c) = \frac{dh_2}{(1-h_2c)^{d+1}}\left[1-\sum_{i=0}^{d} e^{-c_1(1-h_2c)}\left(\frac{\{c_1(1-h_2c)\}^i}{i!}\right)\right] - \frac{c_1^d\, e^{-c_1(1-h_2c)}}{c\,\Gamma(d)} \quad \text{for } c \neq 1/h_2.$$

Proof. The proof follows at once by writing these functions as a sum and then differentiating term by term w.r. to c or by writing them as integrals and applying the Leibnitz rule for partial derivatives.

From Corollary 4.2.3 and Lemma 4.2.1, we get the non-null pdf of the Tiku's statistic U for $\varphi \geq 0$ as

(4.2.6) $$f_U(u) = H\sum_{j=0}^{r_1}(-1)^j\binom{r_1}{j}B(r_2+1,f+j)e^{-u_1} \cdot \sum_{i=0}^{d} u_1^i\{1+h_1(j)u\}^{i-d-1}/i! + H\sum_{j=0}^{r_2}(-1)^j\binom{r_2}{j}B(r_1+1,f+j) \cdot \exp\{-h_2(j)d\varphi\}\{1+h_2(j)u\}^{-d-1} + H\sum_{j=0}^{r_2}(-1)^j\binom{r_2}{j} \cdot \frac{B(r_1+1,f+j)}{\{1-h_2(j)u\}^{d+1}}\left[e^{-h_2(j)d\varphi} - e^{-u_1}\sum_{i=0}^{d}[u_1\{1-h_2(j)u\}]^i/i!\right]$$

for $u > 0$, $u \neq 1/h_2(j)$ $(j = 0,1,\ldots,r_2)$,

where $u_1 = d\varphi/u$. At the singularity points, $u = 1/h_2(j)$ the pdf $f_U(u)$ is taken as zero.

It may be noted that, Khatri's (1981) expression for the pdf of U as given in Section 1.5, differs slightly from the above expression due to some minor integration errors. Also

the derivation of the cdf from the pdf given in equation (4.2.6) is extremly difficult. Consequently, the evaluation of power function using Khatri's approach is not easy, although one may resort to numerical integration techniques. But equation (4.2.5) yields the power function immediately.

Corollary 4.2.3 reduces to the following corollary for $r_1 = r_2 = 0$.

Corollary 4.2.4. The non-null cdf of the KP test is given by

$$(4.2.7) \quad P[U_1 \le c|\varphi] = Q_d(c_1|0) - n_2 \exp(n_1\varphi) Q_d\{c_1| n_1 c/d\}/(n_1+n_2)$$

$$+ n_1 \exp(-n_2\varphi)\, [L_d\{c_1| n_2 c/d\} - \{1+n_2 c/d\}^{-d}]/(n_1+n_2)$$

$$\text{for } 0 \le c < \infty.$$

The cdf agrees with the cdf of the KP test derived by Weinman *et al*. (1973) and Dubey (1973) and given in Section 1.5.

4.3. The LR test statistic.

Let $\Omega = \{\theta_1, \theta_2, \sigma > 0\}$ be the parameter space and $\omega = \{\theta_1 = \theta_2 = \theta,\ \sigma > 0\}$ be the null hypothesis subset of Ω. Let $\{x^{(1)}_{r_1+1}, \ldots, x^{(1)}_{n_1-s_1}, x^{(2)}_{r_2+1}, \ldots, x^{(2)}_{n_2-s_2}\}$ be the set of two independent type II doubly censored samples from $E(\theta_i, \sigma)$ $(i = 1,2)$ with likelihood function

$$L(x^{(1)}_{r_1+1}, \ldots, x^{(1)}_{n_1-s_1}, x^{(2)}_{r_2+1}, \ldots, x^{(2)}_{n_2-s_2}; \theta_1, \theta_2, \sigma)$$

$$= \prod_{i=1}^{2} \frac{n_i!\,\sigma^{-d_i^*}}{r_i!\, s_i!} [1 - \exp\{-\tfrac{1}{\sigma}(x^{(i)}_{r_i+1} - \theta_i)\}]^{r_i} \times$$

$$\cdot \exp\left[-\frac{1}{\sigma}\left\{\sum_{j=r_i+1}^{n_i-s_i}(x_j^{(i)}-\theta_i)+s_i(x_{n_i-s_i}^{(i)}-\theta_i)\right\}\right]$$

$$\text{for } \theta_i \leq x_{r_i+1}^{(i)} \leq \cdots \leq x_{n_i-s_i}^{(i)} \quad (i = 1,2).$$

We here discuss the case $r_1 > 0$, $r_2 > 0$ in detail. Considerably simpler expressions hold for other cases. From equations (2.4.3) and (2.4.4), the ML estimators of θ_1, θ_2 and σ in the parameter space Ω are

$$\hat{\theta}_i = x_{r_i+1}^{(i)} - \hat{\sigma} \log\{n_i/(n_i-r_i)\} \quad (i = 1,2)$$

and $\hat{\sigma} = P_1/d^*$ respectively. Hence, the maximum value of the likelihood function under Ω is

$$(4.3.1) \quad L(\hat{\Omega}) = \frac{\exp(-d^*)}{\hat{\sigma}^{d^*}} \prod_{i=1}^{2} \frac{n_i!}{r_i!\, s_i!} \left(\frac{r_i}{n_i}\right)^{r_i} \left(\frac{n_i-r_i}{n_i}\right)^{(n_i-r_i)}.$$

In the null hypothesis subset ω, the ML estimators of θ and σ for $x_{r_1+1}^{(1)} \leq x_{r_2+1}^{(2)}$ [see, Section 2.6] are given by

$$(4.3.2) \quad \hat{\theta}_o = x_{r_1+1}^{(1)} - \hat{\sigma}_o \log\left\{1 + \frac{r_1 Y}{d^* \hat{\sigma}_o + fY - P}\right\} = x_{r_2+1}^{(2)} - \hat{\sigma}_o \log\left(1 + \frac{r_2 Y}{P - d^* \hat{\sigma}_o}\right),$$

where $\hat{\sigma}_o$ is the solution of the equation

$$e^{Y/\hat{\sigma}_o} = \left[1 + \frac{r_2 Y}{P - d^* \hat{\sigma}_o}\right] \Big/ \left[1 + \frac{r_1 Y}{d^* \hat{\sigma}_o + fY - P}\right]$$

and the notations are same as in Section 2.6.

From equation (4.3.2) we immediately get

$$1 - \exp\left\{-\frac{1}{\hat{\sigma}_o}(x_{r_1+1}^{(1)} - \hat{\theta}_o)\right\} = r_1 Y/\{d^* \hat{\sigma}_o + (f+r_1)Y - P\}$$

and

$$1-\exp\{-\frac{1}{\hat{\sigma}_o}(x^{(2)}_{r_2+1}-\hat{\theta}_o)\} = r_2Y/\{P-d^*\hat{\sigma}_o + r_2Y\}.$$

Further,

$$\frac{1}{\hat{\sigma}_o}\sum_{i=1}^{2}\left[\sum_{j=r_i+1}^{n_i-s_i}(X^{(i)}_j-\hat{\theta}_o) + s_i(X^{(i)}_{n_i-s_i}-\hat{\theta}_o)\right] = \frac{1}{\hat{\sigma}_o}\{P+f(X^{(1)}_{r_1+1}-\hat{\theta}_o)\}.$$

Then the maximum value of the likelihood function under ω is

$$L(\hat{\omega}) = \left[\prod_{i=1}^{2}\frac{n_i!}{r_i!\,s_i!}\right]\left[\frac{r_1Y}{d^*\hat{\sigma}_o+(r_1+f)Y-P}\right]^{r_1}\left[\frac{r_2Y}{P-d^*\hat{\sigma}_o+r_2Y}\right]^{r_2}$$

$$\cdot\left[\frac{d^*\hat{\sigma}_o+fY-P}{d^*\hat{\sigma}_o+(r_1+f)Y-P}\right]^{f}\hat{\sigma}_o^{-d^*}\exp(-P/\hat{\sigma}_o) \text{ for } Y = X^{(2)}_{r_2+1}-X^{(1)}_{r_1+1} \geq 0.$$

Now, the likelihood ratio is given by

$$\lambda = L(\hat{\omega})/L(\hat{\Omega})$$

$$(4.3.3) \qquad = \text{Const.}\left\{\frac{\hat{\sigma}}{\hat{\sigma}_o}\right\}^{d^*}\frac{Y^{r_1+r_2}(d^*\hat{\sigma}_o+fY-P)^f\exp(-P/\hat{\sigma}_o)}{\{d^*\hat{\sigma}_o+(r_1+f)Y-P\}^{r_1+f}\{P-d^*\hat{\sigma}_o+r_2Y\}^{r_2}}$$

for $Y \geq 0$,

where $\text{Const.} = \exp(d^*).\prod_{i=1}^{2} n_i^{n_i}(n_i-r_i)^{-(n_i-r_i)}$.

For $Y = X^{(2)}_{r_2+1}-X^{(1)}_{r_1+1} < 0$, λ is obtained by replacing n_1, n_2, r_1, r_2 by n_2, n_1, r_2, r_1 respectively in equation (4.3.3). The LR test then rejects H_o if $\lambda \leq \lambda_\alpha$, where λ_α is chosen so that $P[\lambda \leq \lambda_\alpha | H_o] = \alpha$.

4.4. Critical points of the tests V_1, V_2, U and λ.

4.4.1. Exact critical points of the test statistics. The critical point c_α of V is obtained by solving the equation

$$(4.4.1) \qquad P\left[V \geq c_\alpha | H_o\right] = \alpha,$$

where α is the chosen level of significance. From equation (4.2.1) it is clear that, c_α is either the solution of

$$(4.4.2) \quad H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j) \left[\{1+h_1(j)(q-c_\alpha)\}^{-d} - \{1+h_1(j)(q+c_\alpha)\}^{-d}\right]/dh_1(j) = 1-\alpha$$

(case in which $c_\alpha \leq q$) or the solution of

$$(4.4.3) \quad H\left[\sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j) \{1+h_1(j)(q+c_\alpha)\}^{-d}/dh_1(j) + \sum_{j=0}^{r_2} (-1)^j \binom{r_2}{j} B(r_1+1, f+j)\{1-h_2(j)(q-c_\alpha)\}^{-d}/dh_2(j)\right] = \alpha$$

(case in which $c_\alpha \geq q$) according as $P_q \geq 1-\alpha$, or $P_q \leq 1-\alpha$ respectively, where

$$P_q = P\left[V \leq q | H_o\right] = H \sum_{j=0}^{r_1} (-1)^j \binom{r_1}{j} B(r_2+1, f+j) \cdot \left[1-\{1+2h_1(j)q\}^{-d}\right]/dh_1(j).$$

The value of c_α can thus be obtained by first evaluating P_q and then applying Newton-Raphson method to the relevant equation. Unlike the one-sided test statistic discussed in Chapter III,

here it is not possible to get a compact expression for c_α even for $r_1 = r_2 = 0$ case.

Now, the critical points $c_\alpha^{(1)}, c_\alpha^{(2)}$ and $c_\alpha^{(3)}$ of the tests V_1, V_2 and U are nothing but the solution of the equation (4.4.1) with $q = q_1, q_2$ and 0 respectively. It is clear from the equation (4.2.1), that the critical point $c_\alpha^{(3)}$ of the test U is simply the solution of the equation (4.4.3) with $q = 0$. Some of these critical points are tabulated in Table 4.4.1 for studying the performance of these satistics.

Since the LR test statistic λ given in equation (4.3.3) is very complicated, we have not attempted to derive the distribution of λ. However, we have obtained simulated critical points λ_α based on 10,000 iterations, for studying the performance of the statistic. These are tabulated in Table 4.4.1. For $r_1 = r_2 = 0$, the tabulated critical points are exact.

In Section 4.5, we have compared the performance of these statistics. It is very difficult to compare the power functions of the test statistics. But Tables 4.5.1, 4.5.2 and some other numerical calculations show that, in between V_1 and V_2 there is very little difference of the power values. However, V_1 is slightly better than V_2. Hence we are suggesting a method for obtaining an approximate value c_α^* for $c_\alpha^{(1)}$. This c_α^* can also be taken as an initial value for solving equation (4.4.1) for $c_\alpha^{(1)}$. An identical approach can be used for $c_\alpha^{(2)}$ and $c_\alpha^{(3)}$.

4.4.2. <u>Approximated critical point c_α^* of V_1.</u> In Section 3.4, we studied an approximate null distribution of T. In the present case, let $Y_1 = T - q_1$ so that $V_1 = |Y_1|$. Then from equation (3.4.7) we have

$$(4.4.4) \quad E(Y_1) = q_1/(d-1), \mathrm{Var}(Y_1) = d^2\left[(d-1)A + q_1^2\right]/\left[(d-1)^2(d-2)\right]$$

with $q_1 \equiv B$. Now, it is clear from Section 3.4 that $Y_1/\sqrt{A}$ has a Student's t distribution with d DF. We improve it slightly by considering two constants $a(> 0)$ and b such that mean and variance of the random variable $a(Y_1+b)$ is equal to the exact mean and variance of a Student's t random variable with d DF. That is,

$$E\left[a(Y_1+b)\right] = 0 \text{ and } \mathrm{Var}\left[a(Y_1+b)\right] = d/(d-2).$$

Then on using equation (4.4.4) we get

$$(4.4.5) \qquad a = (d-1)/\left[d\{(d-1)A+q_1^2\}\right]^{1/2}, \quad b = -q_1/(d-1).$$

Since
$$P\left[V_1 \geq c_\alpha^{(1)} | H_o\right] = 1 - P\left[|Y_1| \leq c_\alpha^{(1)} | H_o\right]$$
$$= 1 - P\left[a(-c_\alpha^{(1)}+b) \leq a(Y_1+b) \leq a(c_\alpha^{(1)}+b)\right]$$
$$\cong 2P\left[T_d \geq t_\alpha\right],$$

$$(4.4.6) \qquad c_\alpha^{(1)} \cong t_\alpha/a - b = c_\alpha^*,$$

where t_α is the $(\alpha/2)$<u>th</u> percentile point of Student's t distribution with d DF. Some of the exact critical points c_α [solution of the equation (4.4.1)] and its approximated value c_α^* [given in the equation (4.4.6)] are tabulated in Table 4.4.2

for different combinations of sample sizes and censoring patterns. Note that, for very small values of d and $|(r_1/n_1 - r_2/n_2)| > 0.1$, the approximation is not satisfactory.

Remark 4.4.1. Similar to the approach discussed in Section 3.4, we also studied a normal approximation for the critical points of V_1. But these are not as good as the Student's t approximated values for most of the cases. Hence, these values are not tabulated.

4.5. Power of the tests V_1, V_2, U and λ.

The power of the test V for testing $H_o : \theta_1 = \theta_2$ against $H_2 : \theta_1 \neq \theta_2$ is given by

$$(4.5.1) \qquad P\,[V \geq c_\alpha|\varphi] = \begin{cases} 1-P_1(c_\alpha|\varphi) & \text{for } \varphi \geq 0 \\ \\ 1-P_2(c_\alpha|\varphi) & \text{for } \varphi < 0, \end{cases}$$

where c_α is the exact critical point, $P_1(c_\alpha|\varphi)$ is given by equation (4.2.4) and $P_2(c_\alpha|\varphi)$ is obtained by interchanging n_1, n_2, r_1, r_2 by n_2, n_1, r_2, r_1 in equation (4.2.4) and evaluating it for $|\varphi|$.

The power functions of V_1, V_2 and U are obtained from equation (4.5.1) by replacing (q, c_α) by $(q_1, c_\alpha^{(1)})$, $(q_2, c_\alpha^{(2)})$ and $(0, c_\alpha^{(3)})$ respectively. Some of these power values are tabulated in Tables 4.5.1 and 4.5.2.

The power of the LR test statistic λ is given by $P\,[\lambda \leq c_\alpha|\varphi]$. As we had mentioned in Section 4.4, the distribution theory of λ is very complicated. Hence we obtained the power

by applying Monte-Carlo technique, using 1000 iterations. However, for $r_1 = r_2 = 0$, the exact power was obtained by using the power function expression as given by Paulson (1941) [see, also Weinman et al. 1973].

For the following sets of sample sizes, censoring patterns, and various values of φ we studied the comparative performance of all the four test statistics for $\alpha = 0.05$.

(i) $n_1=10$, $n_2=8$, $d=14$, $(r_1, r_2)=(0,0)$ and $(r_1, r_2)=(1,1)$ in Table 4.5.1.

(ii) $n_1=n_2=15$, $r_1=1, r_2=3$, $d = 18$ and $d = 24$ in Table 4.5.2.

To high light the shape of the power functions, three such curves for $n_1 = n_2 = 15$, $r_1 = 1$, $r_2 = 3$ and $d = 24$ are drawn in Figure 4.5.1.

Since the difference in power values of V_1 and V_2 is very small, we have not drawn the power function of V_2 in Figure 4.5.1. In all cases studied, it was observed that the "average" power of V_1 was always greater than the "average" power of V_2, where the average was taken of the two values corresponding to $+\varphi$ and $-\varphi$.

A normal approximation similar to the one studied in Section 3.5, for power function is also considered. Since $Y_1 = T-q_1$, the power of the test V_1 is given by

$$P\,[V_1 \geq c_\alpha^{(1)} \mid \varphi] = P[\,|Y_1| \geq c_\alpha^{(1)}\,]$$

$$= P\,[Y_1 \geq c_\alpha^{(1)}] + P\,[Y_1 \leq -c_\alpha^{(1)}]$$

$$= P\left[W_1 \geq 0\right] + P\left[W_2 \leq 0\right], \tag{4.5.2}$$

where $W_1 = X^{(1)}_{r_1+1} - X^{(2)}_{r_2+1} - (q_1 + c^{(1)}_{\alpha})\sigma^*$ and $W_2 = X^{(1)}_{r_1+1} - X^{(2)}_{r_2+1} - (q_1 - c^{(1)}_{\alpha})\sigma^*$.

Similar to the results obtained in Section 3.5, we see that W_i $(i = 1, 2)$ has an asymptotic normal distribution with

$$E(W_1) = (\varphi - c^{(1)}_{\alpha})\sigma \equiv p_1\sigma,$$

$$Var(W_1) = \{A + (q_1 + c^{(1)}_{\alpha})^2/d\}\sigma^2 \equiv g_1\sigma^2,$$

$$E(W_2) = (\varphi + c^{(1)}_{\alpha})\sigma \equiv p_2\sigma$$

and

$$Var(W_2) = \{A + (q_1 - c^{(1)}_{\alpha})^2/d\}\sigma^2 \equiv g_2\sigma^2.$$

Hence from equation (4.5.2), we immediately get

$$P\left[V_1 \geq c^{(1)}_{\alpha} \mid \varphi\right] \cong \Phi(\rho_1/\sqrt{g_1}) + \Phi(-p_2/\sqrt{g_2}). \tag{4.5.3}$$

The exact power values obtained from equation (4.5.1) and its approximated values given in equation (4.5.3) are tabulated in Table 4.5.3 for $\alpha = 0.05$ and selected values of n_1, n_2, r_1, r_2 and d.

Using the power function as a base, the following conclusions may be drawn.

(a) All the three tests V_1, V_2 and U are biased. However, the extent of the bias is different for different tests.

(b) There is a considerable loss of power for all tests even with mild censoring on the left as is evident from Table 4.5.1. But the loss of power is negligible with variations in censoring on the right.

(c) The test statistic U proposed by Tiku (1981) is more biased than V_1 and V_2. In general the statistic U shows poor performance as $|(r_1/n_1-r_2/n_2)|$ increases. Further, the bias of U seems to increase with increase in d.

(d) Table 4.5.1 shows that the test statistic V_2 is relatively more biased than V_1.

(e) We do not have sufficient evidence to conclude that, the LR test statistic λ is unbiased for $r_1 > 0$ and/or $r_2 > 0$, since some simulated values are less than α as in Table 4.5.1. However, for $r_1 = r_2 = 0$, Dubey (1973) established that, the LR test statistic is unbiased [see, also Khatri 1974].

(f) Table 4.5.1, Table 4.5.2 and Figure 4.5.1 show that there is very little difference in the power values of λ and V_1. Since the statistic λ given in equation (4.3.4) is very complicated while the statistic V_1 given in equation (4.1.3) is considerably simple, we strongly recommend the use of test statistic V_1 in such situations.

(g) The normal approximation for the power function is fairly good if $|(r_1/n_1-r_2/n_2)| = D$ (say) is small, even for small values of n_1 and n_2. However for large values of D, the approximation is not that good even with moderately large values of n_1 and n_2. This is well illustrated in Table 4.5.3, for the case $n_1 = 15$, $n_2 = 25$ and $d = 28$. For $(r_1, r_2) = (0,4)$, the maximum difference (among all tabulated values) between approximate and exact values is 0.0274 for $\varphi = 0.30$, whereas for $(r_1, r_2)=(4,0)$, the maximum difference is 0.0566 for $\varphi = -.40$.

TABLE 4.4.1. Critical points $c_\alpha^{(1)}, c_\alpha^{(2)}, c_\alpha^{(3)}$ and λ_α of the tests V_1, V_2, U and λ respectively for $\alpha = 0.05$.

n_1	n_2	r_1	r_2	d	$c_\alpha^{(1)}$	$c_\alpha^{(2)}$	$c_\alpha^{(3)}$	λ_α
10	8	0	0	14	0.3769	0.3825	0.3825	0.2386
10	8	1	1	14	0.5589	0.5654	0.5760	0.0861*
10	10	1	1	16	0.4827	0.4827	0.4827	0.1104*
15	15	1	3	18	0.4049	0.4069	0.5239	0.1576*
15	15	1	3	24	0.3929	0.3947	0.5103	0.2063*
20	15	4	3	26	0.4194	0.4223	0.4223	0.0589*

*Simulated values based on 10000 samples for each sample size.

TABLE 4.4.2. Exact critical point $c_{\alpha}^{(1)}$ in top row and its approximated value c_{α}^{*} given in equation (4.4.6) in bottom row for $\alpha = 0.05$.

d / (r_1, r_2)	$n_1=n_2=10$				$n_1=10,\ n_2=20$			
	4	8	12	16	4	8	12	16
(0,0)	.4459	.3634	.3403	.3295	.3507	.2822	.2630	.2541
	.4535	.3488	.3219	.3096	.3536	.2725	.2522	.2431
(0,1)	.5959	.4672	.4320	.4156	.3875	.3128	.2919	.2821
	.6493	.4714	.4265	.4062	.3972	.3052	.2815	.2707
(0,2)	.8014	.5914	.5351	.5094	.4432	.3499	.3241	.3121
	.9059	.6170	.5456	.5135	.4670	.3468	.3162	.3023
(1,0)	.5959	.4672	.4320	.4156	.5591	.4167	.3776	.3596
	.5753	.4396	.4063	.3914	.5332	.3937	.3607	.3462
(1,1)	.6692	.5370	.5000	.4827	.5430	.4227	.3895	.3741
	.6779	.5214	.4811	.4629	.5333	.4066	.3756	.3618
(2,0)	.8014	.5914	.5351	.5094	.8113	.5679	.5010	.4701
	.7485	.5496	.5027	.4820	.7393	.5224	.4722	.4502

d / (r_1, r_2)	$n_1=n_2=15$				$n_1=15, n_2=25$			
	10	12	16	20	10	16	22	28
(4,4)	.5948 .5819	.5770 .5637	.5556 .5423	.5432 .5301	.4919 .4709	.4527 .4383	.4361 .4250	.4270 .4178
(4,3)	.5575 .5359	.5401 .5198	.5193 .5010	.5073 .4903	.4897 .4645	.4453 .4296	.4269 .4154	.4168 .4078
(4,2)	.5315 .5034	.5127 .4874	.4904 .4689	.4775 .4584	.4924 .4615	.4408 .4233	.4196 .4078	.4082 .3995
(4,1)	.5169 .4827	.4945 .4650	.4681 .4447	.4532 .4334	.5001 .4616	.4394 .4191	.4145 .4020	.4011 .3927
(4,0)	.5149 .4718	.4863 .4512	.4529 .4275	.4340 .4142	.5119 .4642	.4414 .4168	.4119 .3976	.3960 .3873
(3,4)	.5575 .5561	.5401 .5363	.5193 .5131	.5073 .4999	.4299 .4159	.4001 .3888	.3874 .3775	.3803 .3715
(2,4)	.5315 .5422	.5127 .5191	.4904 .4921	.4775 .4768	.3799 .3720	.3551 .3466	.3444 .3360	.3384 .3302
(1,4)	.5169 .5385	.4945 .5107	.4681 .4782	.4532 .4598	.3405 .3397	.3161 .3114	.3057 .2996	.2999 .2932
(C,4)	.5149 .5435	.4863 .5098	.4529 .4705	.4340 .4482	.3110 .3192	.2818 .2830	.2695 .2681	.2628 .2599

TABLE 4.5.1. Exact power of the tests V_1, V_2, U and exact or simulated power of λ for $\alpha = 0.05$, $n_1=10$, $n_2=8$ and $d = 14$.

φ	$r_1=r_2=0$			$r_1=r_2=1$			
	V_1	$V_2 \equiv U$	λ	V_1	V_2	U	$\lambda^{(*)}$
−1.00	.9977	.9982	.9971	.9388	.9490	.9561	.923
−0.80	.9833	.9871	.9795	.8089	.8323	.8498	.779
−0.60	.8927	.9132	.8739	.5580	.5907	.6169	.535
−0.50	.7619	.7982	.7308	.4099	.4404	.4657	.403
−0.40	.5524	.5987	.5158	.2759	.3001	.3205	.254
−0.30	.3194	.3572	.2915	.1729	.1894	.2036	.155
−0.25	.2249	.2536	.2042	.1346	.1476	.1588	.124
−0.20	.1545	.1743	.1403	.1045	.1143	.1229	.083
−0.15	.1064	.1192	.0972	.0818	.0888	.0950	.078
−0.10	.0755	.0832	.0700	.0655	.0701	.0741	.057
−0.05	.0575	.0612	.0549	.0550	.0573	.0594	.052
0.00	.0500	.0500	.0500	.0500	.0500	.0500	.048
0.05	.0523	.0483	.0553	.0504	.0479	.0457	.041
0.10	.0661	.0567	.0729	.0566	.0512	.0464	.049
0.15	.0955	.0782	.1079	.0696	.0605	.0525	.074
0.20	.1472	.1130	.1679	.0905	.0769	.0648	.076
0.25	.2276	.1827	.2586	.1209	.1017	.0843	.106
0.30	.3358	.2746	.3761	.1620	.1362	.1125	.173
0.35	.4602	.3879	.5052	.2147	.1815	.1505	.221
0.40	.5341	.5090	.6278	.2782	.2376	.1988	.288
0.50	.7320	.7243	.8121	.4235	.3765	.3239	.455
0.60	.8964	.8631	.9125	.5356	.5312	.4723	.613
0.80	.9787	.9710	.9821	.8258	.7892	.7443	.835
1.00	.9957	.9941	.9964	.9403	.9242	.9023	.941

*Simulated power based on 1000 samples for each sample size.

TABLE 4.5.2. Exact power of the tests V_1, V_2, U and simulated power of the test λ for $\alpha = 0.05$, $n_1=n_2=15$, $r_1=1$ and $r_2=3$.

φ	d=18				d=24			
	V_1	V_2	U	$\lambda^{(*)}$	V_1	V_2	U	$\lambda^{(*)}$
-1.00	.9960	.9964	.9970	.997	.9973	.9970	.9967	.994
-0.90	.9857	.9871	.9921	.987	.9919	.9924	.9949	.989
-0.80	.9632	.9650	.9782	.950	.9752	.9765	.9861	.973
-0.70	.9128	.9163	.9435	.890	.9336	.9367	.9596	.896
-0.60	.8177	.8234	.8702	.768	.8461	.8517	.8966	.779
-0.50	.6700	.6773	.7421	.573	.6978	.7058	.7741	.588
-0.40	.4863	.4938	.5641	.386	.5045	.5130	.5910	.396
-0.30	.3073	.3134	.3724	.250	.3143	.3212	.3877	.219
-0.20	.1703	.1741	.2130	.112	.1710	.1754	.2189	.113
-0.15	.1221	.1249	.1536	.103	.1216	.1248	.1567	.105
-0.10	.0864	.0883	.1077	.072	.0864	.0884	.1092	.064
0.00	.0500	.0500	.0500	.053	.0500	.0500	.0500	.056
0.05	.0474	.0463	.0338	.058	.0485	.0472	.0336	.066
0.10	.0563	.0541	.0237	.069	.0596	.0566	.0233	.093
0.15	.0814	.0764	.0185	.131	.0869	.0815	.0180	.173
0.20	.1262	.1180	.0179	.191	.1358	.1269	.0172	.223
0.30	.2832	.2725	.0331	.374	.3097	.2931	.0319	.418
0.40	.5143	.4955	.0343	.618	.5430	.5244	.0837	.669
0.50	.7206	.7054	.1927	.795	.7451	.7310	.1973	.825
0.60	.8597	.8503	.3593	.904	.8753	.8670	.3768	.926
0.70	.9353	.9314	.5507	.953	.9445	.9404	.5793	.967
0.80	.9732	.9709	.7213	.937	.9771	.9752	.7524	.988
0.90	.9893	.9884	.8462	.995	.9910	.9902	.8707	.993
1.00	.9960	.9959	.9229	.995	.9966	.9963	.9388	.999

*Simulated power based on 1000 samples for each sample size.

TABLE 4.5.3. Exact and approximate power values of the test statistic V_1 for $\alpha = 0.05$.

$(n_1=n_2=10, d=16)$

ψ	$r_1=r_2=1$		$r_1=0, r_2=1$	
	Exact	Approx.	Exact	Approx.
-1.00	.9786	.9832	.9938	.9956
-0.90	.9551	.9568	.9841	.9851
-0.80	.9098	.9038	.9606	.9577
-0.70	.8298	.8140	.9092	.8990
-0.60	.7045	.6851	.8109	.7960
-0.50	.5331	.5284	.6555	.6475
-0.40	.3593	.3672	.4624	.4721
-0.30	.2110	.2271	.2813	.3021
-0.25	.1560	.1709	.2094	.2291
-0.20	.1143	.1253	.1526	.1675
-0.15	.0844	.0905	.1102	.1186
-0.10	.0647	.0663	.0801	.0825
-0.05	.0536	.0521	.0605	.0590
0.00	.0500	.0474	.0500	.0478
0.05	.0536	.0521	.0481	.0490
0.10	.0647	.0663	.0558	.0634
0.15	.0844	.0905	.0762	.0925
0.20	.1143	.1253	.1145	.1377
0.25	.1560	.1709	.1774	.1996
0.30	.2110	.2271	.2675	.2776
0.40	.3598	.3672	.4967	.4683
0.50	.5381	.5284	.7066	.6672
0.60	.7045	.6851	.8456	.8275
0.70	.8298	.8140	.9250	.9273
0.80	.9098	.9038	.9649	.9755
0.90	.9551	.9568	.9840	.9934
1.00	.9786	.9832	.9929	.9936

TABLE 4.5.3. Contd.

(n_1=15, n_2=25, d=28)

φ	r_1=4, r_2=0		r_1=0, r_2=4	
	Exact	Approx.	Exact	Approx.
-0.60	.8719	.8717	.9963	.9913
-0.55	.8199	.8041	.9739	.9735
-0.50	.7522	.7185	.9546	.9527
-0.45	.6673	.6181	.9141	.9065
-0.40	.5657	.5091	.8443	.8332
-0.35	.4511	.3996	.7412	.7306
-0.30	.3314	.2979	.6075	.6034
-0.25	.2185	.2108	.4588	.4641
-0.20	.1264	.1424	.3179	.3291
-0.15	.0662	.0941	.2032	.2136
-0.10	.0399	.0648	.1222	.1269
-0.05	.0384	.0526	.0729	.0715
0.00	.0500	.0552	.0500	.0464
0.05	.0701	.0714	.0507	.0521
0.10	.0984	.1003	.0796	.0931
0.15	.1362	.1418	.1538	.1758
0.20	.1847	.1957	.2913	.3012
0.25	.2448	.2614	.4722	.4577
0.30	.3166	.3371	.6490	.6216
0.35	.3985	.4202	.7879	.7659
0.40	.4875	.5070	.8814	.8732
0.45	.5792	.5935	.9377	.9403
0.50	.6634	.6757	.9689	.9758
0.55	.7499	.7500	.9851	.9916
0.60	.8199	.8142	.9931	.9975

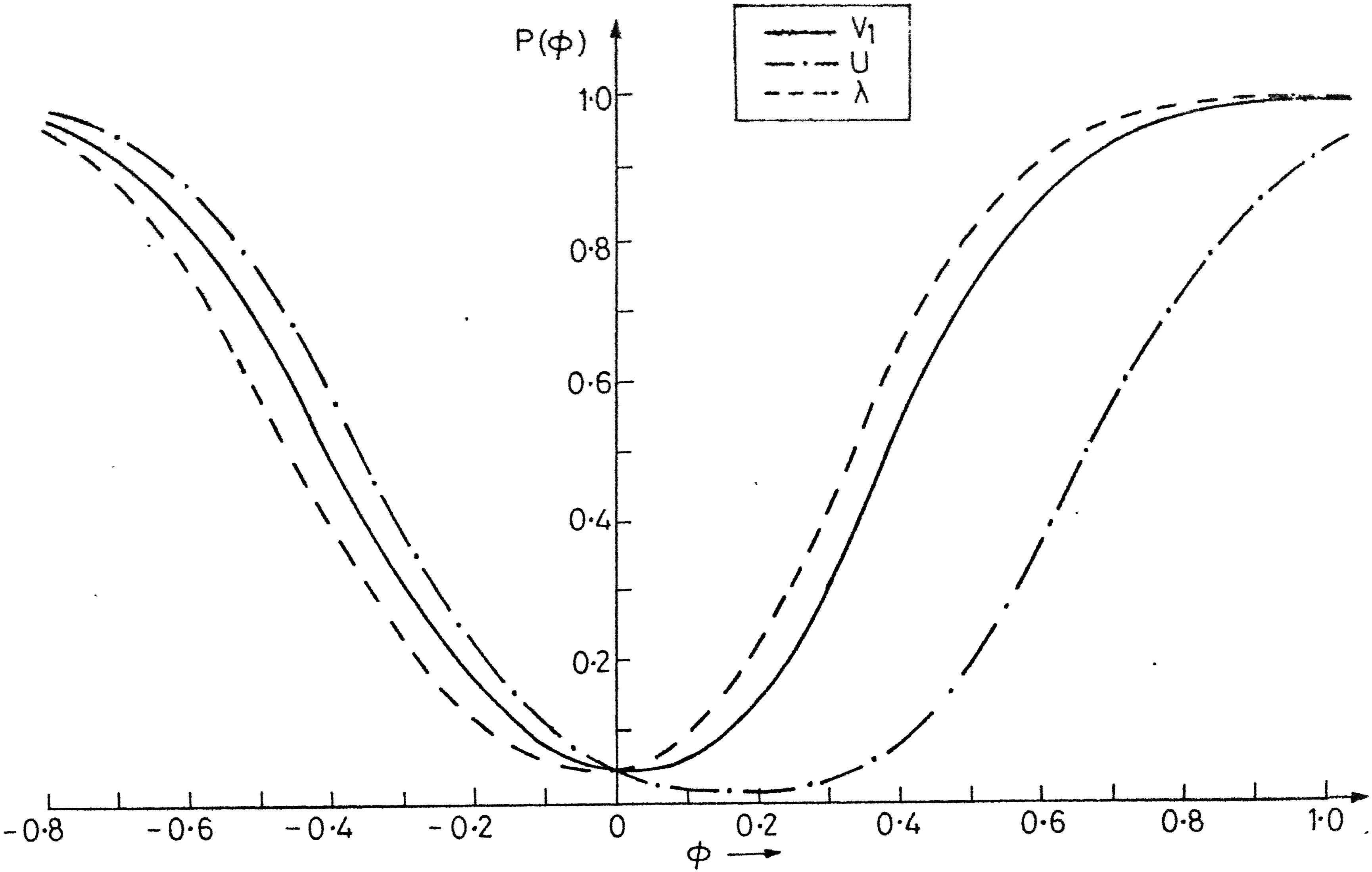

FIGURE 4·5·1. Power functions of V_1, U and λ for $n_1=n_2=15$ $r_1=1, r_2=3$ and $d=24$

CHAPTER V

GENERALIZED STATISTICS FOR K RIGHT CENSORED SAMPLES

5.1. Introduction.

So far we considered the case of testing the equality of location parameters of two exponential distributions. In this chapter, we propose two test statistics for testing $H_o : \theta_1 = \theta_2 = \cdots = \theta_K = \theta$ against the alternative hypotheses $H_1 : \theta_1 > \max_{2\leq j\leq K} (\theta_j)$ and H_2 : at least one θ_i $(i = 1, 2, \cdots, K)$ is different from θ, based on $K(\geq 3)$ independent right censored samples. The neccssary distribution theory of the proposed test statistics is discussed. Some critical points and power values are tabulated. Finally, we compare the performance of these statistics with the test statistics proposed by Khatri (1974) and Singh (1983).

Let $X_1^{(i)}, X_2^{(i)}, \cdots, X_{n_i-s_i}^{(i)}$ $(i = 1, 2, \cdots, K)$ be K independent samples from $E(\theta_i, \sigma)$, where $n_i - s_i \geq 1$. For simplicity of notations, let $X_i = X_1^{(i)}$ $(i = 1, 2, \cdots, K)$ and $X_{(1)} = \min(X_1, X_2, \cdots, X_K)$, that is, X_i is the minimum of the ith sample and $X_{(1)}$ is the minimum of all the observations.

Khatri (1974) derived the LR test for testing H_o against H_2. It is given by

$$(5.1.1) \qquad U_1 = \sum_{i=1}^{K} n_i (X_i - X_{(1)})/d\sigma^*,$$

where

(5.1.2) $$d\sigma^* = \sum_{i=1}^{K} \{ \sum_{j=1}^{n_i-s_i} X_j^{(i)} - n_i X_i + s_i X_{n_i-s_i}^{(i)} \}, \quad d = \sum_{i=1}^{K} (n_i - s_i - 1)$$

are same as in earlier chapters. He obtained the power function of the LR test and showed that

(5.1.3) $$P[U_1 \geq c \mid H_0] = \{B(d, K-1)\}^{-1} \int_{c}^{\infty} y^{K-2}(1+y)^{-K-d+1} \, dy.$$

From equation (5.1.3), it is easy to show that $dU_1/(K-1)$ has an F-distribution with $2(K-1)$ and $2d$ DF, which is denoted by $F_{2(K-1), 2d}$. He also derived two union intersection test statistics from two different view points. These are given by

(5.1.4) $$U_2 = [\max\{n_1(x_1 - x_{(1)}), n_2(x_2 - x_{(1)}), \ldots, n_K(x_K - x_{(1)})\}]/d\sigma^*$$

and

(5.1.5) $$U_3 = [\max\{n_2(x_2 - x_1), \ldots, n_K(x_K - x_1), n_1(x_1 - x_2), \ldots, \ldots, n_1(x_1 - x_K)\}]/d\sigma^*.$$

In all three cases, the test procedure is to reject H_0 if $U_i \geq c_\alpha^{(i)}$, otherwise accept H_0, where the constant $c_\alpha^{(i)}$ ($i=1,2,3$) is determined by solving the equation $P[U_i \geq c_\alpha^{(i)} \mid H_0] = \alpha$. Khatri (1974) also obtained the null distributions of U_2 and U_3, and their power functions under the assumptions $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_K$, which can be achieved by renaming the populations. He has provided some critical points of U_2 and U_3. However, no power comparison studies for U_1, U_2 and U_3 have been done.

Singh (1983) also discussed the LR test procedure for testing H_o against H_2. The test procedure is equivalent to rejecting H_o if $U_4 > c_\alpha^{(4)}$, where

$$U_4 = \sum_{i=1}^{K} n_i(X_i - X_{(1)})/\{(K-1)\sigma^*\} \text{ and } P\,[U_4 \geq c_\alpha^{(4)} | H_o] = \alpha.$$

He has shown that, the distribution of U_4 is $F_{2(K-1),2d}$, but he did not study the power function of U_4. Note that, $U_4 \equiv dU_1/(K-1)$ and the power function of U_1 has been already given by Khatri (1974).

Although there are number of tests for testing H_o against H_2 based on some theoretical considerations as described above, we propose another test based on

$$T_2 = \{\max_{1\leq i\leq K} X_i - \min_{1\leq i\leq K} X_i\}/\sigma^*.$$

This is mainly for comparing the performance of various tests. Further, a generalization of T_2 for left censoring is easier than that of the other tests. Note that, if all samples are of equal sizes, then T_2 is equivalent to U_2.

5.2. Test statistics and their null distributions.

Similar to the one-sided test statistics of the two-sample case, we propose

$$T_1 = \{X_1 - \min(X_2, X_3, \ldots, X_K)\}/\sigma^* \tag{5.2.1}$$

for testing H_o against H_1. Similarly, for testing H_o against H_2, we propose

$$(5.2.2) \qquad T_2 = \{ \max_{1\le i\le K} X_i - \min_{1\le i\le K} X_i\}/\sigma^*.$$

The test procedure is to reject H_o if $T_i \geq c_i$, where $P[T_i \geq c_i | H_o] = \alpha$ $(i = 1, 2)$, and α is the chosen level of significance. For obtaining the critical points c_1 and c_2, we need the distributions of T_1 and T_2 under H_o. These are obtained from the following lemmas :

Lemma 5.2.1. If $W_1, W_2, \ldots, W_K$ are K independent random variables with pdf of W_i given by

$$(5.2.3) \qquad f_{W_i}(w) = n_i \exp(-n_i w), \; w \geq 0 \quad (i = 1, 2, \ldots, K),$$

then the pdf of $Z = W_1 - \min(W_2, \ldots, W_K)$ is

$$(5.2.4) \qquad f(z) = \begin{cases} n_1(N-n_1) \exp\{(N-n_1)z\}/N, & z \leq 0 \\ n_1(N-n_1) \exp\{-n_1 z\}/N, & z > 0, \end{cases}$$

where $N = n_1+n_2+\cdots+n_K$.

Proof. Let $Y = \min(W_2, W_3, \ldots, W_K)$. Then the cdf of Y is given by

$$P[Y \leq y] = 1 - \prod_{i=2}^{K} \exp(-n_i y) = 1-\exp(-N_2 y), \text{where } N_2 = \sum_{i=2}^{K} n_i.$$

Consequently, the pdf of Y is

$$f(y) = N_2 \exp(-N_2 y), \; y \geq 0.$$

The result now follows, on applying Lemma 3.2.1.

Lemma 5.2.2. Let W_i $(i = 1, 2, \ldots, K)$ be K independent random variates with pdf given by equation (5.2.3). Then the pdf of

$Z = \max_{1\le i\le K} W_i - \min_{1\le i\le K} W_i$ is

$$(5.2.5) \quad g(z) = \frac{1}{N} \sum_{i=1}^{K} \sum_{j=1, j\neq i}^{K} n_i n_j e^{-n_j z} \prod_{h=1, h\neq i, j}^{K} \{1-\exp(-n_h z)\}, z > 0,$$

where $N = n_1+n_2+\dots+n_K$.

Proof. Since W_i $(i = 1, 2, \dots, K)$ are K independent random variates with pdf $f_{W_i}(w)$, hence, the cdf of Z is [see, David 1981, p. 26] given by

$$(5.2.6) \quad G(z) = \sum_{i=1}^{K} \int_{-\infty}^{\infty} f_{W_i}(w) \prod_{j=1, j\neq i}^{K} \{F_{W_j}(w+z)-F_{W_j}(w)\} dw,$$

where for $w \geq 0$, $F_{W_j}(w) = 1-\exp(-n_j w)$ is the cdf of W_j.

Substituting for $f_{W_i}(w)$ and $F_{W_j}(w)$, we have

$$G(z) = \sum_{i=1}^{K} \int_{0}^{\infty} n_i e^{-n_i w} \prod_{j=1, j\neq i}^{K} \{e^{-n_j w} - e^{-n_j (w+z)}\} dw$$

$$= \sum_{i=1}^{K} \int_{0}^{\infty} n_i e^{-Nw} \prod_{j=1, j\neq i}^{K} \{1-e^{-n_j z}\} \; dw$$

$$= \sum_{i=1}^{K} \frac{n_i}{N} \prod_{j=1, j\neq i}^{K} \{1-e^{-n_j z}\}$$

$$(5.2.7) \qquad = \sum_{i=1}^{K} G_i(z),$$

where $G_i(z) = \frac{n_i}{N} \prod_{j=1, j\neq i}^{K} \{1-\exp(-n_j z)\} \quad (i = 1, 2, \dots, K).$

Now, $\log G_i(z) = \text{Const.} + \sum_{j=1, j\neq i}^{K} \log \{1-\exp(-n_j z)\}$.

Differentiating w.r. to z, on both the sides of the above equation, we have

$$\frac{g_i(z)}{G_i(z)} = \sum_{j=1, j\neq i}^{K} [n_j \exp(-n_j z)/\{1-\exp(-n_j z)\}], \text{where } g_i(z) = \frac{\partial G_i(z)}{\partial z}$$

Hence, $g_i(z) = \frac{n_i}{N} \sum_{j=1, j\neq i}^{K} n_j \exp(-n_j z) \prod_{h=1, h\neq i, j}^{K} \{1-\exp(-n_h z)\}$.

From equation (5.2.7), the pdf of Z is given by

$$g(z) = \sum_{i=1}^{K} g_i(z)$$

which gives the required equation (5.2.6).

Theorem 5.2.1. Under the null hypothesis H_o, the statistic T_1 defined in equation (5.2.1), has the following cdf :

$$(5.2.8) \quad P[T_1 \leq c|H_o] = \begin{cases} n_1\{1-(N-n_1)c/d\}^{-d}/N, & c < 0 \\ 1-(N-n_1)(1+n_1c/d)^{-d}/N, & c \geq 0, \end{cases}$$

where $N = \sum_{i=1}^{K} n_i$ and $d = \sum_{i=1}^{K} (n_i-s_i-1)$.

Proof. The random variable $W_i = (X_i-\theta_i)/\sigma$ $(i = 1, 2, \ldots, K)$ has the pdf given in equation (5.2.3). Consequently, $Z = W_1 - \min_{2\leq i\leq K} W_i$ follows the distribution as given in equation (5.2.4). Then under H_o, $T_1 = dZ/W$, where $W = d\sigma^*/\sigma$. Now, applying Theorem 3.2.1 we get the required equation (5.2.8).

For general K and n_i, the distribution theory of the statistic T_2 is very complicated, although it is possible to

follow the same approach. We therefore consider the simplifying assumption $K = 3$ in Theorem 5.2.2.

Theorem 5.2.2. For $K = 3$, the null pdf of T_2 is given by

$$(5.2.9) \quad f(t_2) = \frac{1}{N} \sum_{i=1}^{3} n_i(N-n_i)\{(1+n_i t_2/d)^{-d-1} - (1+\sum_{h=1, h\neq i}^{3} n_h t_2/d)^{-d-1}\}$$

for $t_2 \geq 0$,

where $N = n_1+n_2+n_3$.

Proof. Note that, $W_i = (X_i-\theta_i)/\sigma$ $(i = 1,2,3)$ has the pdf given in equation (5.2.3). From Lemma 5.2.2, for $K = 3$, the pdf of $Z = \max(W_1,W_2,W_3) - \min(W_1,W_2,W_3)$ can be rewritten as

$$g(z) = \frac{1}{N} \sum_{i=1}^{3} n_i(N-n_i)\{\exp(-n_i z)-\exp(-\sum_{h=1, h\neq i}^{3} n_h z)\}, \quad z \geq 0.$$

Since $T_2 = dZ/W$, where $W = d\sigma^*/\sigma$, we obtain the desired pdf of T_2 by proceeding on lines similar to that of the proof of Theorem 3.2.1.

If $n_1 = n_2 = \cdots = n_K = n$, then the statistic T_2 given in equation (5.2.2) is equivalent to T_3, where

$$(5.2.10) \qquad T_3 = nT_2 \equiv dU_2.$$

For equal sample sizes, we therefore use T_3. The null distribution of T_3 can be obtained as before. It can also be derived from the cdf of U_2 as given by Khatri (1974). Note that, Khatri has used v/s for U_2 and p for d. Then by using Khatri's expression we have

$$P[T_3 \le c|H_o] = P[U_2 \le c/d|H_o]$$

$$(5.2.11) \qquad = 1-\sum_{j=0}^{K-2}(-1)^j\binom{K-1}{j+1}\{1+(j+1)c/d\}^{-d}, \quad c \ge 0$$

For studying the performance of these statistics, the non-null distributions are derived in the next section. For $K > 3$, the expressions for power functions are very complicated. Hence, we have only considered the case $K = 3$ in detail. Without loss of generality, we have assumed $\theta_1 \ge \theta_2 \ge \theta_3$, since this can be achieved by relabeling the sample.

5.3. Non-null distributions of the statistics.

The non-null distribution of T_1 is derived by using the following lemma:

Lemma 5.3.1. Let W_1, W_2 and W_3 be independent random variates with pdf of W_i given by

$$(5.3.1) \quad f_{W_i}(w) = n_i \exp\{-n_i(w-\varphi_i)\}, \; w > \varphi_i \quad (i = 1,2,3),$$

where $\varphi_1 \ge \varphi_2 \ge \varphi_3$. Then the pdf of $Z = W_1 - \min(W_2, W_3)$ is

$$(5.3.2) \quad f(z) = \begin{cases} (n_2+n_3)b_1 \exp\{(n_2+n_3)z\}, & -\infty < z \le \alpha_2 \\ n_1 b_2 \exp(-n_1 z) + n_3 b_3 \exp(n_3 z), & \alpha_2 < z \le \alpha_3 \\ n_1(b_2+b_4) \exp(-n_1 z), & \alpha_3 \le z < \infty, \end{cases}$$

where $\alpha_j = (\varphi_1-\varphi_j)$ $(j = 2,3)$, $b_1 = n_1 \exp(-n_2\alpha_2-n_3\alpha_3)/(n_1+n_2+n_3)$, $b_2 = n_1 n_2 \exp(n_1\alpha_2-n_3\alpha_3+n_3\alpha_2)/\{(n_1+n_3)(n_1+n_2+n_3)\}$, $b_3 = n_1 \exp(-n_3\alpha_3)/(n_1+n_3)$ and $b_4 = n_3 \exp(n_1\alpha_3)/(n_1+n_3)$.

Proof. The cdf of $Y = \min(W_2, W_3)$ is

$$P[Y \leq y] = 1 - \{1 - P[W_2 \leq y]\}\{1 - P[W_3 \leq y]\}.$$

From this, the pdf of Y is given by

$$f(y) = \begin{cases} n_3 \exp\{-n_3(y-\varphi_3)\}, & \varphi_3 \leq y \leq \varphi_2 \\ (n_2+n_3)\exp\{-(n_2+n_3)y+(n_2\varphi_2+n_3\varphi_3)\}, & \varphi_2 \leq y < \infty. \end{cases}$$

Now, from the jpdf of Y and W_1, and making a transformation $z = w_1 - y$ and $w = w$, we get the marginal pdf of Z as

$$(5.3.3)\quad f(z) = \begin{cases} \int_{\varphi_1-z}^{\infty} p_1(z,y)dy, & -\infty < z \leq \alpha_2 \\ \int_{\varphi_2}^{\infty} p_1(z,y)dy + \int_{\varphi_1-z}^{\varphi_2} p_2(z,y)dy, & \alpha_2 \leq z \leq \alpha_3 \\ \int_{\varphi_2}^{\infty} p_1(z,y)dy + \int_{\varphi_3}^{\varphi_2} p_2(z,y)dy, & \alpha_3 \leq z < \infty, \end{cases}$$

where $p_1(z,y) = n_1(n_2+n_3)\exp(-n_1 z + \sum_{i=1}^{3} n_i\varphi_i - \sum_{i=1}^{3} n_i y)$

and $p_2(z,y) = n_1 n_3 \exp\{-n_1 z + n_1\varphi_1 + n_3\varphi_3 - (n_1+n_3)y\}$.

Equation (5.3.3) now gives the required equation (5.3.2) on simplification.

Theorem 5.3.1. For $\theta_1 \geq \theta_2 \geq \theta_3$, the non-null distribution of T_1 is given by

$$P[T_1 \leq c|\underset{\sim}{\theta}] = P(c|\underset{\sim}{\theta})$$

$$(5.3.4) \quad = \begin{cases} b_1 \{1-(n_2+n_3)c/d\}^{-d}, & c < 0 \\ Q_d(c_3|0) + b_1 L_d\{c_2|(n_2+n_3)c/d\} & \\ -b_3 [L_d(c_2|n_3c/d) - L_d(c_3|n_3c/d)] & \\ -[b_2Q_d\{c_2|n_1c/d\} + b_4Q_d\{c_3|n_1c/d\}], & c \geq 0, \end{cases}$$

where $c_j = d\alpha_j/c$, $\alpha_j = (\theta_1-\theta_j)/\sigma$ $(j = 2, 3)$, $Q_d(\cdot|\cdot)$, $L_d(\cdot|\cdot)$ are given in equation (3.1.2) and b_i $(i = 1, 2, 3, 4)$ are given in Lemma 5.3.1.

<u>Proof</u>. Let $W_i = X_i/\sigma$ $(i = 1, 2, 3)$. Then W_i's are independent random variates with pdf given as in equation (5.3.1) with $\varphi_i = \theta_i/\sigma$. The pdf of $Z = W_1 - \min(W_2, W_3)$ is given in Lemma 5.3.1. Clearly, T_1 can be written as dZ/W, where $W = d\sigma^*/\sigma$. Now, using similar arguments as in Theorem 3.3.1, the cdf of T_1 upto the point c is the integral of the joint density of (T_1, W) over the shaded region shown in Figure 5.3.1.

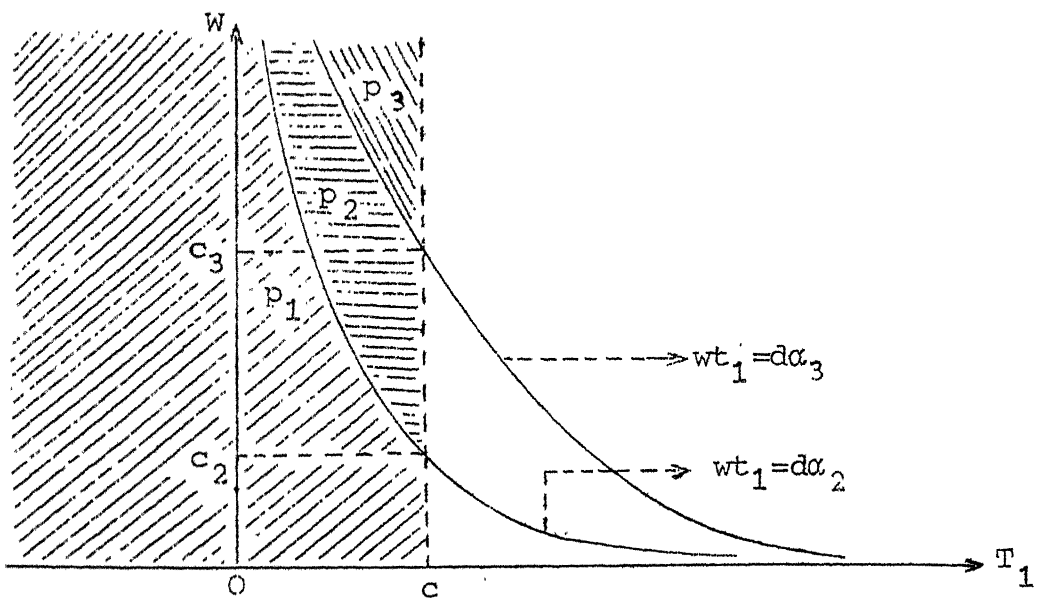

FIGURE 5.3.1. Region showing the cdf of T_1 upto the point c.

Consequently,

$$P\,[T_1 \le c|\underset{\sim}{\theta}] = \int_0^\infty (\int_{-\infty}^{c} p_1 dt_1)\, dw = J_1 \text{ (say) for } c < 0$$

and

$$P\,[T_1 \le c|\underset{\sim}{\theta}] = \int_0^\infty (\int_{-\infty}^{0} p_1 dt_1) dw + \int_0^{c_2} (\int_0^{c} p_1 dt_1) dw$$

$$+ \int_{c_2}^{\infty} (\int_0^{d\alpha_2/w} p_1 dt_1) dw + \int_{c_2}^{c_3} (\int_{d\alpha_2/w}^{c} p_2 dt_1) dw$$

$$+ \int_{c_3}^{\infty} (\int_{d\alpha_2/w}^{d\alpha_3/w} p_2 dt_1) dw + \int_{c_3}^{\infty} (\int_{d\alpha_3/w}^{c} p_3 dt_1) dw$$

$$= J_2+J_3+J_4+J_5+J_6+J_7 \quad \text{(say)} \quad \text{for } c \ge 0,$$

where $p_1 = (n_2+n_3)b_1 \exp[-w\{1-(n_2+n_3)t_1/d\}]\, w^d/d!$,

$p_2 = n_1 b_2 \exp\{-w(1+n_1 t_1/d)\}\, w^d/d! + n_3 b_3 \exp\{-w(1+n_3 t_1/d)\}\, w^d/d!$

and

$p_3 = n_1(b_2+b_4) \exp\{-w(1+n_1 t_1/d)\}\, w^d/d!$.

Simplifications for J_j's ($j = 1,2,\ldots,7$) are similar to that of I_i's of Theorem 3.3.1. Their simplified forms are as follows :

$J_1 = b_1 \{1-(n_2+n_3)c/d\}^{-d}$

$J_2 = b_1$

$J_3 = b_1 [L_d\{c_2|(n_2+n_3)c/d\} - 1 + Q_d(c_2|0)]$

$J_4 = b_1 \exp\{(n_2+n_3)\alpha_2-1\}\, Q_d(c_2|0)$

$$J_5 = b_1 [\exp(-n_1\alpha_2)\{Q_d(c_2|0)-Q_d(c_3|0)\} - \{Q_d(c_2|\,n_1 c/d)$$
$$- Q_d(c_3|n_1 c/d)\}] + b_3 [\,L_d(c_3|n_3 c/d)$$
$$-L_d(c_2|n_3 c/d) - \exp(n_3\alpha_2)\{Q_d(c_2|0)-Q_d(c_3|0)\}$$

$J_6 = b_2\{\exp(-n_1\alpha_2)-\exp(-n_1\alpha_3)\}Q_d(c_3|0)+b_3\{\exp(n_3\alpha_3)-\exp(n_3\alpha_2\}Q_d(c_3|0)$

$J_7 = b_4\ [\exp(-n_1\alpha_3)Q_d(c_3|0)-Q_d\{c_3|n_1c/d\}]$.

By combining these expressions, we get the required cdf of T_1 given in equation (5.3.4).

For unequal sample sizes, the non-null distribution of T_2 is very complicated. It can be obtained either by proceeding as in Theorem 5.3.1 or by the method given by Khatri (1974). For equal sample size case $n_1 = n_2 = n_3 = n$ and $\theta_1 \geq \theta_2 \geq \theta_3$ the non-null distribution of T_3 obtained in a similar manner or derived from Khatri's result is given by

$$(5.3.5) \quad P\ [T_3 \leq c|\underset{\sim}{\theta})] = Q_d(c_3|0) - 2g_1\{(1+c/d)^{-d}-(1+2c/d)^{-d}\}$$
$$-(g_6+g_7)Q_d(c_2|c/d)+2g_7Q_d(c_2|2c/d)$$
$$-(g_4+g_5)Q_d(c_3|c/d)+2g_3Q_d(c_3|2c/d)$$
$$-(g_1+g_2)L_d(c_2|c/d)+2g_1\ L_d(c_2|2c/d)$$
$$+(g_2-g_1)\ L_d(c_3|c/d), \qquad c \geq 0,$$

where $c_j = d\gamma_j/c$, $\gamma_j = n(\theta_1-\theta_j)/\sigma \quad (j = 2,3)$,

$g_1 = \exp(-\gamma_2-\gamma_3)/6$, $g_2 = \exp\ (-\gamma_3)/2$, $g_3 = \exp(2\gamma_3-\gamma_2)/6$,

$g_4 = \exp(\gamma_3)/2$, $g_5 = \exp\ (\gamma_3-\gamma_2)/2$, $g_6 = \exp\ (\gamma_2-\gamma_3)/2$,

$g_7 = \exp\ (2\gamma_2-\gamma_3)/6$, and d, $Q_d(\cdot|\cdot)$ and $L_d(\cdot|\cdot)$ are given in equation (3.1.2).

One of the objectives of this chapter is to study the performance of the statistic T_1 for different combinations of

n_1, n_2, n_3 and d along with a comparative study of the statistics T_1, T_3, U_3 and U_4 for $n_1 = n_2 = n_3$. Methods of obtaining the required critical points for all these test procedures are given in the next section.

5.4. The critical points of the test statistics.

From equation (5.2.8), the upper 100α percent critical point $c_{1,\alpha}$ of the test statistic T_1 is given by

$$(5.4.1) \quad c_{1,\alpha} = \begin{cases} [1 - \{\frac{n_1}{(1-\alpha)N}\}^{1/d}] \frac{d}{(N-n_1)}, & n_1/N \geq 1-\alpha \\ [(\frac{N-n_1}{\alpha N})^{1/d} - 1] \frac{d}{n_1}, & n_1/N \leq 1-\alpha, \end{cases}$$

where α is the chosen level of significance. Some critical points of T_1 are tabulated in Table 5.4.1.

The critical point $c_{3,\alpha}$ of the test statistic T_3 is given by $P\,[T_3 \geq c_{3,\alpha} | H_o] = \alpha$. From equation (5.2.11), it is clear that, $c_{3,\alpha}$ is the solution of the equation

$$(5.4.2) \qquad \sum_{j=0}^{K-2} (-1)^j \binom{K-1}{j+1} \{1+(j+1)c_{3,\alpha}/d\}^{-d} = \alpha.$$

As we had mentioned in earlier sections, the statistics T_3 and U_2 are very closely related to each other. Although Khatri (1974) has provided a table for the critical points of U_2, he has not mentioned the procedure for solving equation (5.4.2). Here we suggest Newton-Raphson method for obtaining $c_{3,\alpha}$ by solving the equation (5.4.2) with an approximate critical point as the initial

value. Note that, for d not too small, $\{1+(j+1)c_{3,\alpha}/d\}^{-d}$ decreases rapidly as j increases. Hence, the approximate critical point $c^*_{3,\alpha}$ may be taken as the solution of the equation (5.4.2) corresponding to the term j = 0. Consequently,

$$(5.4.3) \qquad c^*_{3,\alpha} = d\{(\frac{K-1}{\alpha})^{1/d} - 1\}.$$

It is clear from the equation (5.4.2), that for K = 2, $c^*_{3,\alpha}$ given in equation (5.4.3) is the exact critical point.

Some exact and approximate critical points are tabulated in Table 5.4.2 for $\alpha = 0.05$ and for some selected values of K and d. From the extensive study made in this direction, we conclude that, except for large K and very small d, the approximation is reasonably good. Note that, $c_{3,\alpha}/d$ is equal to the critical point of U_2 tabulated by Khatri (1974).

For the statistic U_3, Khatri (1974) has tabulated the critical points for $\alpha = 0.05$, 0.01 and for some selected values of n_1, n_2, n_3 and d. For equal sample size case, the critical point is the solution of

$$(5.4.4) \quad \sum_{j=0}^{K-2}(-1)^j\binom{K}{j+2}(1+u+ju)^{-d} + \sum_{j=0}^{K-2}(-1)^j\binom{K-1}{j+1}(1+u+2ju)^{-d} = K\alpha.$$

As for T_3, the solution of this equation for terms corresponding to j = 0, namely

$$(5.4.5) \qquad (K-1)(K+2)(1+u)^{-d} = 2K\alpha$$

gives a fairly good approximation to the exact critical point for large values of d. This could be used as an initial value for solving equation (5.4.4). Table 5.4.3 gives exact and approximate critical points of U_3 for $\alpha = 0.05$ and selected values of K and d.

As Khatri (1974) and Singh (1983) have shown that U_4 has an $F_{2(K-1),2d}$-distribution, the critical points of U_4 can be obtained easily.

Some critical points of these test statistics T_1, T_3, U_3 and U_4 are tabulated in Table 5.4.1 for K = 3, $n_1=n_2=n_3=n$, $\alpha = 0.05$ and $\alpha = 0.10$. These are used for studying the power function of these tests.

5.5. Power of the tests.

The power of the test T_1 is $P[T_1 > c_{1,\alpha}|\underset{\sim}{\theta}]$, where $c_{1,\alpha}$ is the exact critical point given in equation (5.4.1). By making use of the non-null distribution of T_1 given by equation (5.3.4), some power values of $\theta_1 \geq \theta_2 \geq \theta_3$ are evaluated and tabulated in Tables 5.5.1, 5.5.2 and 5.5.3 for $\alpha = 0.05$ and different combinations of n_1, n_2, n_3 and d.

Table 5.5.1 shows that, for fixed values of n_2, n_3 and d, the power of the test increases very rapidly as n_1 increases. But this is not the case for changes in n_2, n_3 and d for fixed n_1, as is seen in Tables 5.5.2 and 5.5.3.

The power of the test T_3 is obtained by using equation (5.3.5). Since the derivation of the power functions of T_1 and T_3 involved lengthy calculations, the simulated power values of these tests are also tabulated along with the exact values in Table 5.5.4. This serves as a check for theoretical expressions.

The power functions expressions of the tests U_3 and U_4 provided by Khatri (1974) are extremely complicated even for $n_1 = n_2 = n_3$ case. Hence, only the simulated power values are obtained by using 1000 iterations. Some of these values are tabulated in Table 5.5.4. The nature of T_1 is entirely different from the remaining statistics. It can be used for testing the equality of θ_i's $(i = 1, 2, \ldots, K)$ against a specified alternative $\theta_1 > \max(\theta_2, \theta_3, \ldots, \theta_K)$ (with suitable relabelling if necessary). It can be seen from Table 5.5.4, that the power of T_1 is considerably higher than that of other three statistics.

Since the power values for all the tests are provided only for $\theta_1 \geq \theta_2 \geq \theta_3$, it is difficult to compare the performance of these statistics. However, some conclusions can be drawn from Table 5.5.4. Both tests T_3 and U_3 perform equally well. The test U_3 performs slightly better than T_3 when $(\theta_1-\theta_2)$ is large, while T_3 performs better than U_3 if $(\theta_1-\theta_2)$ is small. If $(\theta_1-\theta_2)$ is small, then U_4 is better than T_3 while for $(\theta_1-\theta_2)$ large, the reverse is the case. Similar conclusions are expected for other values of K. The LR test statistic U_4 is recommended for testing H_0 against a general

alternative hypothesis, since its critical points are easy to evaluate from the F-distribution even for unequal sample sizes. Against a specified alternative like $\theta_1 > \max(\theta_2, \theta_3, \ldots, \theta_K)$, the statistic T_1 is recommended.

TABLE 5.4.1. Exact critical points of the tests T_1, T_3, U_3 and U_4 for $K = 3$ and $n_1 = n_2 = n_3 = n$.

Tests	n = 11, d = 30		n = 21, d = 60	
	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.05$	$\alpha = 0.10$
T_1	0.2459	0.1780	0.1261	0.0913
T_3	3.9036	3.1125	3.7878	3.0402
U_3	0.1236	0.0977	0.0600	0.0477
U_4	2.52	2.04	2.45	1.99

TABLE 5.4.2. Exact critical point $c_{3,\alpha}$ of T_3 in top row and its approximated value $c^*_{3,\alpha}$ as given in equation (5.4.3) in bottom row for $\alpha = 0.05$.

d \ K	2	3	4	5	6	8	10
2	6.9443 6.9443	9.6651 10.6491	11.4509 13.4919	12.7893 15.8885	13.8618 18.0000	15.5267 21.6643	16.8016 24.8328
3	5.1433 5.1433	6.8972 7.2599	8.0316 8.7446	8.8785 9.9266	9.5566 10.9248	10.6096 12.5775	11.4168 13.9386
4	4.4590 4.4590	5.8589 6.0595	6.7533 7.1326	7.4181 7.9628	7.9494 8.6491	8.7736 9.7592	9.4053 10.6514
5	4.1028 4.1028	5.3222 5.4564	6.0937 6.3397	6.6648 7.0112	7.1203 7.5594	7.8258 8.4337	8.3662 9.1262
6	3.8853 3.8853	4.9959 5.0959	5.6931 5.8716	6.2073 6.4547	6.6166 6.9266	7.2497 7.6722	7.7340 8.2 57
7	3.7389 3.7389	4.7770 4.8567	5.4245 5.5638	5.9006 6.0909	6.2789 6.5149	6.8631 7.1804	7.3095 7.6987
8	3.6337 3.6337	4.6201 4.6867	5.2321 5.3462	5.6809 5.8349	6.0369 6.2262	6.5860 6.8373	7.0050 7.3108
9	3.5546 3.5546	4.5022 4.5597	5.0876 5.1845	5.5159 5.6453	5.8552 6.0129	6.3777 6.5848	6.7761 7.0261
10	3.4928 3.4928	4.4104 4.4613	4.9751 5.0597	5.3875 5.4992	5.7137 5.8489	6.2156 6.3913	6.5978 6.8084
15	3.3158 3.3158	4.1479 4.1821	4.6537 4.7076	5.0205 5.0893	5.3094 5.3903	5.7518 5.8529	6.0874 6.2052
20	3.2317 3.2317	4.0235 4.0510	4.5016 4.5436	4.8469 4.8991	5.1181 5.1785	5.5322 5.6057	5.3455 5.9295
50	3.0873 3.0873	3.8106 3.8234	4.2415 4.2667	4.5501 4.5798	4.7909 4.8239	5.1565 5.1941	5.4314 5.4722
100	3.0411 3.0411	3.7427 3.7578	4.1585 4.1793	4.4554 4.4795	4.6867 4.7129	5.0368 5.0658	5.2993 5.3302

TABLE 5.4.3. Exact critical point of U_3 in top row and its approximated value given by equation (5.4.5) in bottom row for $\alpha = 0.05$.

d \ K	2	3	4	5	6	8	10
2	3.4721 3.4721	4.5099 4.7736	5.1599 5.7082	5.1317 6.4833	5.2622 7.1650	6.0097 8.3541	6.2515 9.3923
3	1.7144 1.7144	2.1559 2.2183	2.4305 2.5569	2.4610 2.8259	2.5454 3.0548	2.8057 3.4395	2.9938 3.7622
4	1.1147 1.1147	1.3776 1.4028	1.5399 1.5900	1.5687 1.7356	1.6260 1.8574	1.7717 2.0584	1.8933 2.2237
5	0.8206 0.8205	1.0031 1.0164	1.1152 1.1411	1.1390 1.2368	1.1812 1.3162	1.2803 1.4457	1.3661 1.5508
6	0.6475 0.6475	0.7858 0.7940	0.8702 0.8860	0.8900 0.9560	0.9230 1.0136	0.9973 1.1070	1.0625 1.1822
7	0.5341 0.5341	0.6447 0.6503	0.7120 0.7226	0.7287 0.7772	0.7556 0.8220	0.8147 0.8942	0.8668 0.9520
8	0.4542 0.4542	0.5461 0.5501	0.6018 0.6093	0.6161 0.6539	0.6389 0.6904	0.6877 0.7488	0.7308 0.7955
9	0.3949 0.3949	0.4733 0.4764	0.5207 0.5265	0.5333 0.5640	0.5528 0.5946	0.5944 0.6435	0.6311 0.6824
10	0.3493 0.3493	0.4175 0.4200	0.4587 0.4633	0.4699 0.4956	0.4369 0.5219	0.5231 0.5639	0.5549 0.5971
15	0.2211 0.2211	0.2623 0.2633	0.2369 0.2339	0.2939 0.3073	0.3044 0.3231	0.3260 0.3473	0.3450 0.3663
20	0.1616 0.1616	0.1910 0.1916	0.2085 0.2096	0.2136 0.2229	0.2211 0.2337	0.2364 0.2505	0.2499 0.2638
50	0.0617 0.0617	0.0725 0.0726	0.0738 0.0791	0.0307 0.0338	0.0835 0.0376	0.0390 0.0935	0.0933 0.0932
100	0.0304 0.0304	0.0356 0.0357	0.0387 0.0333	0.0396 0.0411	0.0409 0.0429	0.0436 0.0457	0.0459 0.0479

TABLE 5.5.1. Exact power of the test T_1 for $\alpha = 0.05$, $n_2=20$, $n_3=15$, $d = 30$, $K=3$ and $\theta_3=0$.

θ_1	θ_2 \ n_1	5	10	15	20	25	30
.00	.00	.0500	.0500	.0500	.0500	.0500	.0500
.05	.00	.0641	.0824	.1059	.1359	.1745	.2240
.10	.00	.0824	.1359	.2241	.3663	.5561	.7207
.15	.00	.1058	.2241	.4624	.7543	.9013	.9497
.20	.00	.1359	.3674	.7696	.9490	.9827	.9913
.25	.00	.1745	.5728	.9409	.9910	.9970	.9985
.30	.00	.2241	.7823	.9886	.9984	.9995	.9997
.05	.05	.0584	.0690	.0824	.0992	.1206	.1477
.10	.05	.0749	.1137	.1743	.2683	.3949	.5180
.15	.05	.0963	.1875	.3630	.5989	.7715	.8663
.20	.05	.1236	.3081	.6474	.8769	.9533	.9762
.25	.05	.1588	.4891	.8763	.9758	.9918	.9959
.30	.05	.2038	.6992	.9712	.9958	.9986	.9993
.35	.05	.2616	.8660	.9946	.9993	.9998	.9999
.10	.10	.0723	.1074	.1632	.2509	.3694	.4820
.15	.10	.0928	.1771	.3395	.5526	.6954	.7706
.20	.10	.1191	.2909	.6004	.8035	.8919	.9368
.25	.10	.1530	.4611	.8186	.9417	.9779	.9888
.30	.10	.1964	.6596	.9407	.9886	.9961	.9980
.35	.10	.2521	.8268	.9864	.9980	.9993	.9997
.40	.10	.3228	.9301	.9974	.9997	.9999	.9999
.20	.20	.1169	.2845	.5868	.7778	.8503	.8836
.25	.20	.1500	.4506	.7912	.8968	.9320	.9488
.30	.20	.1927	.6425	.9029	.9561	.9759	.9859
.35	.20	.2473	.8018	.9591	.9870	.9951	.9975
.40	.20	.3166	.9027	.9867	.9975	.9991	.9996
.45	.20	.4023	.9572	.9970	.9996	.9999	.9999
.30	.30	.1921	.6411	.8999	.9504	.9666	.9740
.35	.30	.2466	.7995	.9529	.9770	.9848	.9886
.40	.30	.3158	.8989	.9783	.9902	.9946	.9969
.45	.30	.4012	.9517	.9909	.9971	.9989	.9994
.40	.40	.3157	.8986	.9777	.9889	.9925	.9942
.50	.50	.5006	.9768	.9950	.9975	.9983	.9987
.60	.60	.7112	.9948	.9989	.9994	.9996	.9997
.70	.70	.8723	.9988	.9998	.9999	.9999	.9999

TABLE 5.5.2. Exact power of the test T_1 for $\alpha = 0.05$, $n_1 = 15$, $d = 22$, $K = 3$ and $\theta_3 = 0$.

($n_2 = 5$)

θ_1	θ_2 \ n_3	5	10	15	20	25	30
.00	.00	.0500	.0500	.0500	.0500	.0500	.0500
.05	.00	.1058	.1058	.1058	.1058	.1058	.1058
.10	.00	.2234	.2238	.2239	.2239	.2239	.2240
.15	.00	.4284	.4427	.4483	.4520	.4538	.4550
.20	.00	.6364	.6874	.7134	.7283	.7375	.7434
.25	.00	.7784	.8467	.8820	.9021	.9142	.9219
.30	.00	.8656	.9273	.9556	.9701	.9780	.9826
.35	.00	.9185	.9656	.9836	.9913	.9949	.9967
.40	.00	.9505	.9838	.9940	.9975	.9989	.9994
.45	.00	.9700	.9923	.9978	.9993	.9997	.9999
.10	.10	.1511	.1827	.1973	.2053	.2102	.2134
.15	.10	.2960	.3649	.3978	.4161	.4273	.4346
.20	.10	.4738	.5883	.6481	.6828	.7043	.7184
.25	.10	.6449	.7674	.8319	.8688	.8911	.9053
.30	.10	.7790	.8821	.9300	.9547	.9682	.9761
.35	.10	.8656	.9435	.9732	.9860	.9920	.9950
.40	.10	.9185	.9732	.9901	.9959	.9981	.9991
.45	.10	.9505	.9874	.9963	.9988	.9996	.9998
.20	.20	.4299	.5732	.6422	.6803	.7032	.7179
.25	.20	.5646	.7388	.8206	.8639	.8890	.9043
.30	.20	.6803	.8456	.9155	.9485	.9655	.9749
.35	.20	.7846	.9143	.9621	.9815	.9901	.9942
.40	.20	.8659	.9566	.9844	.9938	.9973	.9987
.45	.20	.9185	.9792	.9940	.9981	.9993	.9997
.30	.30	.6537	.8401	.9141	.9482	.9654	.9749
.35	.30	.7359	.9038	.9595	.9808	.9899	.9941
.40	.30	.8061	.9432	.9811	.9930	.9971	.9987
.45	.30	.8694	.9685	.9915	.9975	.9992	.9997
.40	.40	.7900	.9412	.9803	.9929	.9971	.9987
.50	.50	.8726	.9784	.9957	.9990	.9998	.9999

TABLE 5.5.2 Contd:

$(n_2 = 30)$

θ_1	θ_2 \ n_3	5	10	15	20	25	30
.00	.00	.0500	.0500	.0500	.0500	.0500	.0500
.05	.00	.1053	.1058	.1058	.1059	.1059	.1059
.10	.00	.2240	.2240	.2240	.2240	.2240	.2240
.15	.00	.4550	.4558	.4563	.4567	.4570	.4572
.20	.00	.7434	.7474	.7502	.7522	.7536	.7547
.25	.00	.9219	.9269	.9303	.9326	.9343	.9354
.30	.00	.9826	.9854	.9871	.9883	.9890	.9896
.35	.00	.9967	.9977	.9982	.9985	.9987	.9989
.40	.00	.9994	.9997	.9998	.9999	.9999	.9999
.10	.10	.0995	.1315	.1531	.1682	.1790	.1371
.15	.10	.2070	.2722	.3162	.3469	.3690	.3853
.20	.10	.3873	.4914	.5603	.6074	.6406	.6645
.25	.10	.6190	.7209	.7858	.8283	.8570	.8768
.30	.10	.8342	.8908	.9250	.9463	.9598	.9687
.35	.10	.9514	.9711	.9821	.9883	.9921	.9944
.40	.10	.9894	.9945	.9969	.9982	.9989	.9993
.45	.10	.9980	.9991	.9996	.9998	.9999	.9999
.20	.20	.3118	.4573	.5444	.5998	.6369	.6627
.25	.20	.4685	.6534	.7545	.8135	.8498	.8732
.30	.20	.6182	.7966	.8826	.9267	.9505	.9642
.35	.20	.7677	.8954	.9498	.9742	.9858	.9915
.40	.20	.8994	.9597	.9831	.9925	.9965	.9982
.45	.20	.9705	.9894	.9960	.9984	.9993	.9997
.30	.30	.5724	.7841	.8791	.9257	.9502	.9641
.36	.30	.6765	.8705	.9428	.9722	.9852	.9913
.40	.30	.7683	.9250	.9736	.9899	.9957	.9980
.45	.30	.8591	.9615	.9888	.9965	.9988	.9996
.40	.40	.7406	.9204	.9728	.9897	.9957	.9980

TABLE 5.5.3. Exact power of the test T_1 for $\alpha = 0.05$, $n_1 = 30$, $n_2 = 20$, $n_3 = 15$, $K = 3$ and $\theta_3 = 0$.

θ_1	θ_2 \ d	5	10	15	20	25	30
.00	.00	.0500	.0500	.0500	.0500	.0500	.0500
.05	.00	.2129	.2213	.2231	.2237	.2239	.2240
.10	.00	.5463	.6347	.6736	.6959	.7104	.7207
.15	.00	.8182	.9074	.9317	.9416	.9467	.9497
.20	.00	.9438	.9823	.9880	.9898	.9907	.9913
.25	.00	.9854	.9969	.9979	.9982	.9984	.9985
.30	.00	.9966	.9995	.9996	.9997	.9997	.9997
.10	.10	.3773	.4309	.4541	.4673	.4759	.4820
.15	.10	.6486	.7250	.7492	.7603	.7666	.7706
.20	.10	.8465	.9064	.9224	.9297	.9340	.9368
.25	.10	.9475	.9787	.9847	.9870	.9881	.9888
.30	.10	.9852	.9960	.9973	.9977	.9979	.9980
.35	.10	.9964	.9993	.9995	.9996	.9996	.9997
.40	.10	.9992	.9999	.9999	.9999	.9999	.9999
.20	.20	.8088	.8609	.8734	.8787	.8817	.8836
.25	.20	.9097	.9380	.9440	.9465	.9479	.9488
.30	.20	.9635	.9791	.9827	.9843	.9853	.9859
.35	.20	.9879	.9952	.9966	.9971	.9973	.9975
.40	.20	.9966	.9991	.9994	.9995	.9995	.9996
.45	.20	.9992	.9998	.9999	.9999	.9999	.9999
.30	.30	.9551	.9689	.9717	.9729	.9736	.9740
.35	.30	.9795	.9862	.9875	.9881	.9884	.9886
.40	.30	.9918	.9953	.9961	.9965	.9967	.9969
.45	.30	.9973	.9989	.9992	.9994	.9994	.9994
.40	.40	.9899	.9931	.9937	.9940	.9941	.9942
.50	.50	.9977	.9985	.9986	.9987	.9987	.9987
.60	.60	.9995	.9997	.9997	.9997	.9997	.9997
.70	.70	.9999	.9999	.9999	.9999	.9999	.9999

TABLE 5.5.4. Exact and simulated powers of the tests T_1 and T_3, and simulated powers of the tests U_3 and U_4 for $\alpha = 0.05$, $\theta_3 = 0$ and $n_1=n_2=n_3=n$.

(n = 11, d = 30)

θ_1	θ_2	T_1	T_1^*	T_3	T_3^*	U_3^*	U_4^*
.00	.00	.050	.066	.050	.053	.049	.053
.05	.00	.087	.099	.056	.054	.058	.064
.10	.00	.150	.147	.076	.078	.075	.082
.15	.00	.260	.271	.111	.128	.129	.156
.20	.00	.441	.469	.173	.200	.211	.226
.25	.00	.669	.674	.279	.286	.319	.266
.30	.00	.850	.846	.439	.439	.492	.392
.35	.00	.944	.940	.629	.629	.653	.601
.40	.00	.981	.980	.795	.788	.829	.654
.50	.00	.997	.998	.961	.960	.972	.890
.10	.10	.117	.112	.082	.072	.063	.115
.15	.10	.202	.206	.107	.109	.100	.165
.20	.10	.345	.342	.153	.149	.141	.236
.25	.10	.541	.518	.232	.247	.254	.319
.30	.10	.731	.734	.358	.371	.411	.461
.35	.10	.870	.869	.522	.528	.529	.530
.40	.10	.947	.947	.691	.667	.731	.660
.50	.10	.993	.993	.917	.917	.940	.858
.20	.20	.334	.338	.207	.225	.169	.429
.25	.20	.521	.516	.271	.277	.236	.526
.30	.20	.699	.688	.376	.376	.376	.629
.35	.20	.827	.825	.519	.517	.520	.760
.40	.20	.908	.906	.668	.674	.719	.769
.50	.20	.982	.984	.882	.889	.915	.884
.30	.30	.696	.678	.469	.463	.386	.762

*Simulated powers based on 1000 samples for each sample size.

TABLE 5.5.4. Contd.

(n = 21, d = 60)

θ_1	θ_2	T_1	T_1^*	T_3	T_3^*	U_3^*	U_4^*
.00	.00	.050	.042	.050	.048	.051	.043
.05	.00	.143	.135	.073	.064	.066	.072
.10	.00	.407	.416	.160	.174	.175	.183
.15	.00	.849	.853	.403	.392	.433	.360
.20	.00	.981	.980	.804	.810	.866	.660
.25	.00	.998	.999	.969	.976	.989	.906
.05	.05	.112	.111	.079	.073	.075	.090
.10	.05	.318	.319	.143	.154	.157	.180
.15	.05	.717	.716	.328	.329	.328	.335
.20	.05	.946	.947	.682	.675	.679	.670
.25	.05	.993	.993	.924	.927	.932	.923
.10	.10	.307	.313	.193	.199	.146	.400
.15	.10	.686	.686	.351	.349	.347	.628
.20	.10	.900	.913	.661	.661	.699	.783
.25	.10	.981	.986	.882	.890	.908	.898
.30	.10	.998	.997	.974	.982	.987	.958
.20	.20	.888	.864	.756	.749	.701	.898
.25	.20	.961	.960	.883	.882	.889	.944
.30	.20	.988	.985	.957	.950	.956	.966
.30	.30	.986	.989	.969	.975	.974	.993

*Simulated powers based on 1000 samples for each sample size.

CHAPTER VI

GENERALIZED STATISTICS FOR THE EQUAL SAMPLE CASE WHEN ONE OBSERVATION IS MISSING ON THE LEFT

6.1. Introduction and test statistics.

In Chapter V, tests for the equality of location parameters of K ($\geq$ 3) populations are discussed, when the smallest observation is available in each sample. Here the problem is extended to the case when the smallest observation is missing but the second smallest observation is available in each sample of size n, that is,

$$X_2^{(i)}, X_3^{(i)}, \ldots, X_{n-s_i}^{(i)} \quad (i = 1, 2, \ldots, K)$$

are the available observations from the ith population with $(n-s_i) \geq 2$.

Similar to T_1 of Chapter V, the test V_1 defined by

$$(6.1.1) \qquad V_1 = \{X_2^{(1)} - \min_{2\leq i\leq K} (X_2^{(i)})\}/\sigma^*$$

is proposed for testing $H_o : \theta_1 = \theta_2 = \cdots = \theta_K = \theta$ against $H_1 : \theta_1 > \max_{2\leq i\leq K} (\theta_i)$, where

$$(6.1.2) \quad d\sigma^* = \sum_{i=1}^{K} \{ \sum_{j=2}^{n-s_i} X_j^{(i)} + s_i X_{n-s_i}^{(i)} - (n-1)X_2^{(i)} \}, d = \sum_{i=1}^{K} (n-s_i-2).$$

The generalization of statistics T_2 and U_3 of Chapter V are V_2 and V_3 respectively. These for equal sample

size case are given by

(6.1.3) $$V_2 = \{\max_{1\le i\le K} X_2^{(i)} - \min_{1\le i\le K} X_2^{(i)}\}/\sigma^*$$

and

(6.1.4) $$V_3 = [\max_{2\le j\le K}\{(X_2^{(1)}-X_2^{(j)}),\ (X_2^{(j)}-X_2^{(1)})\}]/\sigma^*.$$

Both of these are proposed for testing H_o against H_2 : atleast one θ_i is different from θ. Compared to two-sample case (K = 2), the LR test is much more complicated even for K = 3. Even the derivation of ML estimates under H_o is far from simple and requires a full study in itself. Consequently, we have not studied the LR test in this case and have left it as an open problem.

The test procedure is to reject H_o if $V_i \geq c_i$, where the constants c_i are obtained from solving the equations

(6.1.5) $$P[V_i \geq c_i | H_o] = \alpha,$$

where α is the chosen level of significance. The required null distribution of these statistics are derived in the next section.

6.2. <u>Distribution theory</u>.

<u>Lemma 6.2.1.</u> Let Y_i $(i = 1, 2, \ldots, K)$ be K i.i.d. random variates with pdf

(6.2.1) $$f(y) = n(n-1)\ [\exp\{-(n-1)y\}-\exp(-ny)],\ y \geq 0,$$

then $Z = \{Y_1-\min(Y_2, Y_3, \ldots, Y_K)\}$ has the pdf given by

$$(6.2.2) \qquad f(z) = \begin{cases} f_1(z), & z < 0 \\ f_2(z), & z \geq 0, \end{cases}$$

where

$$f_1(z) = (K-1)\sum_{j=0}^{K-2}(-1)^j\binom{K-2}{j}n^{K-j}(n-1)^{j+2}\left\{\frac{e^{j_4 z}}{j_1 \cdot j_2} - \frac{e^{j_5 z}}{j_2 \cdot j_3}\right\},$$

$$f_2(z) = (K-1)\sum_{j=0}^{K-2}(-1)^j\binom{K-2}{j}n^{K-j}(n-1)^{j+2}\left\{\frac{e^{-(n-1)z}}{j_1 \cdot j_2} - \frac{e^{-nz}}{j_2 \cdot j_3}\right\},$$

$j_1 = (n-1)K+j$, $j_2 = j_1+1$, $j_3 = j_1+2$, $j_4 = (n-1)(K-1)+j$

and $j_5 = j_4 + 1$.

<u>Proof</u>. From equation (6.2.1), the cdf of Y_i is

$$(6.2.3) \qquad F(y) = [1+(n-1)\exp(-ny)-n\exp\{-(n-1)y\}], \quad y \geq 0.$$

The pdf of $Z_2 = \min(Y_2, Y_3, \ldots, Y_K)$ is then given by

$$f(z_2) = (K-1)\,[ne^{-(n-1)z_2}-(n-1)e^{-nz_2}]^{K-2}\; n(n-1)[e^{-(n-1)z_2}-e^{-nz_2}]$$

$$= (K-1)n^{K-1}(n-1)e^{-(n-1)(K-1)z_2}(1-e^{-z_2})\,[1-(\tfrac{n-1}{n})e^{-z_2}]^{K-2}.$$

Expanding $\{(1-(n-1)\exp(-z_2)/n\}^{K-2}$ as a binomial sum , we get

$$f(z_2) = (K-1)(1-e^{-z_2})\sum_{j=0}^{K-2}(-1)^j\binom{K-2}{j}n^{K-j-1}(n-1)^{j+1}$$

$$\cdot \exp[-\{(n-1)(K-1)+j\}z_2].$$

Now, the jpdf of Z_2 and $Z_1 \equiv Y_1$ is

$$(6.2.4)\quad f(z_2, z_1) = (K-1) \sum_{j=0}^{K-2} (-1)^j \binom{K-2}{j} n^{K-j} (n-1)^{j+2} (1-e^{-z_2})$$

$$\cdot \exp\left[-\{(n-1)(K-1)+j\} z_2\right]$$

$$\cdot \left[\exp\{-(n-1)z_1\} - \exp(-nz_1)\right], \quad z_1, z_2 \geq 0.$$

Make a transformation $z = z_1 - z_2, z_2 = z_2$; then the range of the transformed variables are $z_2 \geq \max(0, -z)$, $-\infty < z < \infty$. Thus, the marginal pdf of Z is

$$(6.2.5)\qquad f(z) = \begin{cases} \int_{-z}^{\infty} f(z_2,\ z+z_2)\, dz_2, & z < 0 \\ \int_{0}^{\infty} f(z_2,\ z+z_2)\, dz_2, & z \geq 0, \end{cases}$$

where $f(\cdot,\cdot)$ is given as in equation (6.2.4). On simplification, the equation (6.2.5) gives the required density function of Z.

<u>Lemma 6.2.2.</u> Let Y_i $(i = 1, 2, \ldots, K)$ be K i.i.d. random variates with pdf given as in equation (6.2.1), then the pdf of

$$Z = \max_{1 \leq i \leq K} (Y_i) - \min_{1 \leq i \leq K} (Y_i) \text{ is}$$

$$(6.2.6)\quad f(z) = K \sum_{j=0}^{K-1} (-1)^{K-j-1} \binom{K-1}{j} n^{j+1} (n-1)^{K-j} \{1-e^{-(n-1)z}\}^{j-1}$$

$$\cdot \{1-e^{-nz}\}^{K-j-2} \{n(K-j-1)e^{-nz} + (n-1)je^{-(n-1)z}$$

$$- (nK-n-j)e^{-(2n-1)z}\}, \quad z \geq 0.$$

<u>Proof.</u> The cdf of Z is (David, 1981, p. 26)

$$(6.2.7)\qquad G(z) = K \int_0^{\infty} f(y) \{F(y+z) - F(y)\}^{K-1}\, dy,$$

where $f(\cdot)$ and $F(\cdot)$ are the pdf and cdf of Y_i given in equations (6.2.1) and (6.2.3) respectively. Substituting for $f(\cdot)$ and $F(\cdot)$ in equation (6.2.7), expanding the term $\{F(y+z)-F(y)\}^{(K-1)}$ as a binomial sum and integrating w.r. to y we get

$$(6.2.8) \qquad G(z) = \sum_{j=0}^{K-1} G_j(z)$$

where

$$(6.2.9) \qquad G_j(z) = K(-1)^{K-j-1}\binom{K-1}{j}n^{j+1}(n-1)^{K-j} \cdot \frac{\{1-e^{-(n-1)z}\}^j\,\{1-e^{-nz}\}^{K-j-1}}{(nK-1-j)(nK-j)} \,.$$

Taking logarithm on both sides of equation (6.2.9) and differentiating with respect to z, we get

$$(6.2.10) \qquad \frac{\partial G_j(z)}{\partial z} = \left[\frac{j(n-1)e^{-(n-1)z}}{\{1-e^{-(n-1)z}\}} + \frac{n(K-j-1)e^{-nz}}{\{1-e^{-nz}\}}\right] G_j(z) .$$

Differentiating both sides of equation (6.2.8) with respect to z, and substituting for $\partial G_j(z)/\partial z$ from equation (6.2.10), we get the required pdf of Z given in equation (6.2.6).

<u>Theorem 6.2.1.</u> The null distribution of the statistic V_1 defined in equation (6.1.1) is given

$$(6.2.11) \qquad f(v_1) = \begin{cases} f_1(v_1) , & v_1 < 0 \\ f_2(v_1) , & v_1 \geq 0, \end{cases}$$

where

$$f_1(v_1) = (K-1)\sum_{j=0}^{K-2}(-1)^j\binom{K-2}{j}n^{K-j}(n-1)^{j+2}\left[\{1-j_4v_1/d\}^{-(d+1)}/(j_1j_2)\right.$$

$$\left.-\{1-j_5v_1/d\}^{-(d+1)}/(j_2j_3)\right],$$

$$f_2(v_2) = (K-1)\sum_{j=0}^{K-2}(-1)^j\binom{K-2}{j}n^{K-j}(n-1)^{j+2}\left[\{1+(n-1)v_1/d\}^{-(d+1)}/(j_1j_2)\right.$$

$$\left.-\{1+nv_1/d\}^{-(d+1)}/(j_2j_3)\right],$$

and j_i $(i = 1,\ldots,5)$ are given in equation (6.2.2).

<u>Proof</u>. Note that, $Y_i = (X_2^{(i)} - \theta_i)/\sigma$ $(i = 1, 2, \ldots, K)$ has the distribution given as in equation (6.2.1). Hence, under H_o, $V_1 = dZ/W$, where $2W = 2d\sigma^*/\sigma$ has a χ^2_{2d} distribution and $Z = Y_1 - \min(Y_2, Y_3, \ldots, Y_K)$ has the pdf given in equation (6.2.2). By writing the jpdf of Z and W, and making a transformation $v_1 = dz/w$, $w = w$, and integrating w.r. to w, we get the required pdf of V_1 given in equation (6.2.11).

For K = 2, equation (6.2.11) simplifies to

$$f(v_1) = \begin{cases} \dfrac{n(n-1)}{2(2n-1)}\left[n\{1-(n-1)v_1/d\}^{-d-1}-(n-1)\{1-nv_1/d\}^{-d-1}\right], & v_1 < 0 \\[2ex] \dfrac{n(n-1)}{2(2n-1)}\left[n\{1+(n-1)v_1/d\}^{-d-1}-(n-1)\{1+nv_1/d\}^{-d-1}\right], & v_1 \geq 0 \end{cases}$$

which agrees with the null distribution of T, given in equation (3.2.7) for $r_1 = r_2 = 1$ and $n_1 = n_2 = n$ case.

For K = 3, the null distribution of V_1 is given by

$$(6.2.12)\quad f(v_1)=\begin{cases} B_1\ [B_2\{1-2(n-1)v_1/d\}^{-d-1}-B_3\{1-(2n-1)v_1/d\}^{-d-1} \\ \qquad +B_4\{1-2nv_1/d\}^{-d-1}],\ v_1<0 \\ B_1\ [B_5\{1+(n-1)v_1/d\}^{-d-1}-B_6\{1+nv_1/d\}^{-d-1}],\ v_1\geq 0, \end{cases}$$

where $B_1 = 2n(n-1)/\{3(3n-1)(3n-2)\}$, $B_2 = n^2(3n-1)$,

$B_3 = 3n(n-1)(2n-1)$, $B_4 = (n-1)^2(3n-2)$,

$B_5 = n(5n-3)$ and $B_6 = (n-1)(5n-2)$.

It is easy to establish the following relations among B_i's :

$$B_2 - B_3 + B_4 = B_5 - B_6 = 4n-2,$$

$$B_1\ \{\frac{B_2}{2n-2} - \frac{B_3}{2n-1} + \frac{B_4}{2n}\} = \frac{1}{3},$$

and
$$B_1\ \{\frac{B_5}{n-1} - \frac{B_6}{n}\} = \frac{2}{3}.$$

These relations can be used for showing that $f(v_1)$ given in equation (6.2.12) is continuous at $v_1 = 0$ and it is indeed a pdf.

Since the null distribution of statistics V_2 and V_3 are complicated, only the case $K = 3$ is considered and discussed in the following theorems :

<u>Theorem 6.2.2.</u> Under H_o, the distribution of V_2 for $K = 3$ is

$$(6.2.13)\quad f(v_2) = 3B_1\ [-B_2\{1+2(n-1)v_2/d\}^{-d-1}+B_3\{1+(2n-1)v_2/d\}^{-d-1}$$
$$-B_4\{1+2nv_2/d\}^{-d-1}+B_5\{1+(n-1)v_2/d\}^{-d-1}$$
$$-\ B_6\{1+nv_2/d\}^{-d-1}],\ v_2\geq 0,$$

where B_i $(i = 1,\ldots,6)$ are given in equation (6.2.12).

Proof. Note that, $Y_i = (X_2^{(i)}-\theta_i)/\sigma$ $(i = 1, 2, 3)$ are i.i.d. random variates with common pdf given in equation (6.2.1). Now, from Lemma 6.2.2, for $K = 3$, the pdf of

$$Z = \max\,(Y_1, Y_2, Y_3) - \min\,(Y_1, Y_2, Y_3)$$

can be written as

$$f(z)=3B_1\ [-B_2 e^{-2(n-1)z}+B_3 e^{-(2n-1)z}-B_4 e^{-2nz}+B_5 e^{-(n-1)z}-B_6 e^{-nz}],\quad z \geq 0.$$

Under H_o, $V_2 = dZ/W$, where $W = d\sigma^*/\sigma$. Now, making the transformation $v_2 = dz/w$, $w = w$, and integrating w.r.to w, we obtain the required pdf of V_2 given in equation (6.2.13).

Theorem 6.2.3. The null cdf of the statistic V_3 for $K = 3$ is

$$(6.2.14)\quad P\,[V_3 \leq c|H_o] = 1+A_1\{1+nc/d\}^{-d}-A_2\{1+(n-1)c/d\}^{-d}$$

$$+ A_3\{1+2nc/d\}^{-d}-A_4\{1+(2n-1)c/d\}^{-d}+A_5\{1+2(n-1)c/d\}^{-d}$$

$$+ A_6\{1+3nc/d\}^{-d}-A_7\{1+(3n-1)c/d\}^{-d}-A_8\{1+(3n-2)c/d\}^{-d}$$

$$+ A_9\{1+3(n-1)c/d\}^{-d},\quad c \geq 0,$$

where $A_1=(n-1)[1+(n-1)/(2n-1)-(n-1)^2/3-n^3/(3n-2)+2n^2(n-1)/(3n-1)]$,

$A_2=n[1+n/(2n-1)-(n-1)^3/(3n-1)-n^2/3+2n(n-1)^2/(3n-2)]$,

$A_3=(n-1)^3/\{3(3n-1)\}$, $A_4=2n^2(n-1)^2/\{(3n-1)(3n-2)\}$,

$A_5=n^3/\{3(3n-2)\}$, $A_6=(n-1)^2[1+2(n-1)/3-2n^2/(3n-1)]$,

$A_7 = 2n(n-1)^2\ [1/(2n-1) - n/(3n-2) + (n-1)/(3n-1)]$,

$A_8 = 2n^2(n-1)\ [1/(2n-1) - n/(3n-2) + (n-1)/(3n-1)]$,

and $A_9 = n^2\ [1 - 2n/3 + 2(n-1)^2/(3n-2)]$.

Proof. Let $Y_i = (X_2^{(i)} - \theta_i)/\sigma$ $(i = 1, 2, 3)$. Note that, Y_i's are i.i.d. random variates with pdf $f(y)$ and cdf $F(y)$ as given in equations (6.2.1) and (6.2.3) respectively.

Let $Z = \max\{(Y_1-Y_2), (Y_1-Y_3), (Y_2-Y_1), (Y_3-Y_1)\}$. Thus,

$$P\ [Z \le z|H_o] = P\ [Y_1-Y_2 \le z, Y_1-Y_3 \le z,\ Y_2-Y_1 \le z,\ Y_3-Y_1 \le z]$$

$$= P\ [Y_2 \ge Y_1-z,\ Y_3 \ge Y_1-z,\ Y_2 \le Y_1+z,\ Y_3 \le Y_1+z]$$

$$= \int_{-\infty}^{\infty} P\ [Y_1-z \le Y_2 \le Y_1+z, Y_1-z \le Y_3 \le Y_1+z | Y_1=y]\ f(y)\ dy$$

$$(6.2.15) \qquad = \int_{-\infty}^{\infty} Q(z,y)\ dy,$$

where $Q(z,y) = f(y) \prod_{j=2}^{3} [F_j(y+z) - F_j(y-z)]$. Since $f(y) = 0$ for $y < 0$, hence,

$$(6.2.16) \qquad P\ [Z \le z|H_o] = \int_0^z Q(z,y)dy + \int_z^{\infty} Q(z,y)\ dy.$$

Note that, the first term of the equation (6.2.16) reduces to $\int_0^z f(y)\{F(y+z)\}^2\ dy$. Now substituting for $f(y)$ and $F(y)$, we get the simplified form of the equation (6.2.16) as

$$(6.2.17) \quad P\ [Z \le z|H_o] = 1 + A_1e^{-nz} - A_2e^{-(n-1)z} + A_3e^{-2nz} - A_4e^{-(2n-1)z}$$
$$+ A_5e^{-2(n-1)z} + A_6e^{-3nz} - A_7e^{-(3n-1)z}$$
$$- A_8e^{-(3n-2)z} + A_9e^{-3(n-1)z}, \quad z \ge 0.$$

Under H_o, $V_3 = dZ/W$, where $2W = 2d\sigma^*/\sigma$ has a χ^2_{2d}-distribution. Thus,

$$(6.2.18) \quad P\left[V_3 \le c | H_o\right] = \int_0^\infty P\left[Z \le cW/d | W=w\right] e^{-w} w^{d-1} dw/(d-1)! \; .$$

Substituting from equation (6.2.17) and simplifying the resulting expression, we get the required null cdf of V_3.

6.3. Moments of the statistics under H_o for $K = 3$.

Note that, the pdf of V_i contains factors like $(1-av)^{-d-1}$ and $(1+av)^{-d-1}$. This allows the evaluation of moments of V_i by using the following lemma :

Lemma 6.3.1. For $a > 0$ and $0 \le h < d$

$$\text{(i)} \quad \int_{-\infty}^{0} v^h (1-av)^{-d-1}\, dv = (-1)^h B(h+1, d-h)/a^{h+1}$$

and

$$\text{(ii)} \quad \int_{0}^{\infty} v^h (1+av)^{-d-1}\, dv = B(h+1, d-h)/a^{h+1}.$$

Proof. Making a substitution $t = (1-av)^{-1}$, we have

$$\int_{-\infty}^{0} v^h (1-av)^{-d-1}\, dv = \int_0^1 \left(\frac{t-1}{a}\right)^h \frac{t^{(d-h-1)}}{a}\, dt$$

$$= (-1)^h \int_0^1 (1-t)^h\, t^{(d-h-1)}\, dt/a^{h+1}$$

$$= (-1)^h B(h+1, d-h)/a^{h+1}.$$

Part (ii) of the lemma can be proved by making the substitution $v = -u$ in part (i).

Now, from Lemma 6.3.1 and equation (6.2.12), the hth moment of V_1 about zero for $K = 3$ and $h < d$ is given by

$$E(V_1^h) = B_1 \cdot B(h+1, d-h) d^{h+1} \left[(-1)^h B_2/(2n-2)^{h+1} - (-1)^h B_3/(2n-1)^{h+1} \right.$$

$$\left. + (-1)^h B_4/(2n)^{h+1} + B_5/(n-1)^{h+1} - B_6/n^{h+1} \right].$$

Expressions for the moments of V_2 and V_3 can be written down in a similar manner.

Remark 6.3.1. It is easy to see from the pdf of V_1 and V_2 given in equations (6.2.12) and (6.2.13) respectively, that

$$E(V_2) = 3\, E(V_1).$$

This can also be justified from the fact that

$$V_2 = W_1 + W_2 + W_3,$$

where $W_i = \{X_2^{(i)} - \min\limits_{1 \leq j \leq 3,\, j \neq i} (X_2^{(j)})\}/\sigma^*$ $(i = 1, 2, 3)$ and $W_1 \equiv V_1$.

6.4. Critical points of the tests for K = 3.

The upper 100α percent critical point c_i of the test statistic V_i is the solution of equation (6.1.5). From equation (6.2.12), we have $F_{V_1}(0) = \frac{1}{3}$ and hence c_1 is either the solution of

$$(6.4.1) \qquad B_1 \left[\frac{B_5}{(n-1)} \{1+(n-1)c_1/d\}^{-d} - \frac{B_6}{n} \{1+nc_1/d\}^{-d} \right] = \alpha$$

or of

$$(6.4.2) \qquad 1 - B_1 \left[\frac{B_2}{2(n-1)} \{1-2(n-1)c_1/d\}^{-d} - \frac{B_3}{(2n-1)} \{1-(2n-1)c_1/d\}^{-d} \right.$$

$$\left. + \frac{B_4}{2n} \{1-2nc_1/d\}^{-d} \right] = \alpha$$

according as $\alpha \leq 2/3$, or $\alpha \geq 2/3$ respectively. From equation (6.2.13), c_2 is the solution of equation

$$(6.4.3) \qquad 3B_1 \Big[-\frac{B_2}{2(n-1)}\{1+2(n-1)c_2/d\}^{-d} + \frac{B_3}{2n-1}\{1+(2n-1)c_2/d\}^{-d}$$

$$-\frac{B_4}{2n}\{1+2n\,c_2/d\}^{-d} + \frac{B_5}{(n-1)}\{1+(n-1)c_2/d\}^{-d}$$

$$-\frac{B_6}{n}\{1+nc_2/d\}^{-d}\Big] = \alpha.$$

Similarly, the critical point c_3 is obtained from the equation (6.2.14).

Some critical points of V_1, V_2 and V_3 are tabulated in Tables 6.4.1, 6.4.2 and 6.4.3 respectively for $\alpha = 0.05$ and $K = 3$.

These are calculated by using Newton Raphson method with approximate critical points as the initial values. Approximate values are obtained by using the fact that the null distribution of $\{V_i-E(V_i)\}/\{Var(V_i)\}^{1/2}$ is approximately normal. This gives the approximate critical points as

$$c^{(i)}_{app.} = c_i^*\{Var(V_i)\}^{1/2} + E(V_i) \qquad (i = 1, 2, 3),$$

where c_i^* is the upper 100α percent of $N(0,1)$. Some numerical calculations show that upper $100\alpha/2$ percent point of $N(0,1)$ distribution gives closer approximation for V_2 and V_3 than the 100α percent point. This may be due to the fact that the distribution of V_1 is in the interval $(-\infty,\infty)$, whereas the distribution of V_2 and V_3 are confined to the interval $(0,\infty)$.

Some exact and approximate critical points for all the three tests are tabulated in Tables 6.4.4 for $\alpha = 0.05$ and $K = 3$. It is clear from the Table 6.4.4, that the approximate critical points are reasonably good for large values of d.

6.5. Performance of the tests.

It does not appear simple to evaluate the non-null distributions of these statistics. Consequently, we use Monte-Carlo techniques for the calculation of power of these tests. Some of these values based on 1000 iterations are tabulated in Table 6.5.1 for $\alpha = 0.05$ and $K = 3$.

On the basis of these calculations, the statistic V_1 is recommended for testing H_0 against a specified alternative like $H_1 : \theta_1 > \max(\theta_2, \theta_3)$. This conclusion is similar to the use of T_1 of Chapter V against H_1. For testing against the alternative H_2, it is observed that V_2 performs better if $\theta_1 - \theta_2$ is very small, otherwise the performance of V_3 is better.

In Table 6.5.2, the power values of the test T_1 (of Chapter V) and V_1 are tabulated for $\alpha = 0.05$, $K = 3$, $n_1=n_2=n_3=n=16$ and $d = 6(12)42$. These calculations show that there is a considerable loss of power due to censoring of the smallest observation. This highlights the importance of smallest observations for testing the equality of location parameters. Similar conclusions were drawn by Greenberg and Sarhan (1962) for the estimation of location parameters.

TABLE 6.4.1. Exact critical points of the test V_1 for $\alpha = 0.05$ and $K = 3$.

n \ d	6	9	15	18	27	3n−6
4	1.3938					
5	1.0721	0.9866				
6	0.8722	0.8026				
7	0.7357	0.6770	0.6340			
8	0.6363	0.5855	0.5483	0.5394		
9	0.5607	0.5159	0.4831	0.4753		
10	0.5012	0.4612	0.4319	0.4249		
11	0.4532	0.4170	0.3905	0.3841	0.3739	
12	0.4135	0.3805	0.3563	0.3506	0.3412	0.3394
13	0.3803	0.3499	0.3277	0.3224	0.3138	0.3107
14	0.3520	0.3239	0.3033	0.2984	0.2904	0.2866
15	0.3277	0.3015	0.2823	0.2778	0.2703	0.2659
16	0.3065	0.2820	0.2640	0.2598	0.2528	0.2480
17	0.2878	0.2648	0.2480	0.2440	0.2375	0.2324
18	0.2714	0.2497	0.2338	0.2300	0.2239	0.2187
19	0.2567	0.2361	0.2211	0.2176	0.2118	0.2065
20	0.2435	0.2240	0.2098	0.2064	0.2009	0.1955

TABLE 6.1.2. Exact critical points of the test V_2 for $\alpha = 0.05$ and $K = 3$.

n \ d	6	9	15	18	27	3n-6
4	2.0681					
5	1.5888	1.4193				
6	1.2919	1.1540				
7	1.0893	0.9729	0.8892			
8	0.9419	0.8413	0.7689	0.7518		
9	0.8299	0.7412	0.6774	0.6623		
10	0.7417	0.6624	0.6054	0.5920		
11	0.6706	0.5989	0.5473	0.5352	0.5155	
12	0.6119	0.5465	0.4994	0.4883	0.4704	0.4669
13	0.5627	0.5025	0.4593	0.4491	0.4326	0.4268
14	0.5209	0.4651	0.4251	0.4157	0.4004	0.3930
15	0.4848	0.4329	0.3957	0.3869	0.3727	0.3642
16	0.4534	0.4049	0.3701	0.3618	0.3486	0.3394
17	0.4259	0.3803	0.3476	0.3398	0.3274	0.3177
18	0.4015	0.3585	0.3276	0.3204	0.3086	0.2987
19	0.3797	0.3391	0.3099	0.3030	0.2919	0.2818
20	0.3602	0.3217	0.2940	0.2874	0.2769	0.2667

TABLE 6.4.3. Exact critical points of the test V_3 for $\alpha = 0.05$ and $K = 3$.

n \ d	6	9	15	18	27	3n−6
4	1.9390					
5	1.4895	1.3349				
6	1.2111	1.0852				
7	1.0211	0.9149	0.8387			
8	0.8830	0.7911	0.7252	0.7097		
9	0.7779	0.6970	0.6389	0.6252		
10	0.6953	0.6229	0.5710	0.5588		
11	0.6286	0.5632	0.5162	0.5052	0.4873	
12	0.5736	0.5139	0.4710	0.4610	0.4447	0.4415
13	0.5275	0.4726	0.4332	0.4239	0.4089	0.4036
14	0.4882	0.4374	0.4009	0.3923	0.3785	0.3718
15	0.4544	0.4071	0.3732	0.3652	0.3523	0.3446
16	0.4250	0.3808	0.3490	0.3415	0.3295	0.3211
17	0.3992	0.3576	0.3278	0.3208	0.3094	0.3007
18	0.3763	0.3371	0.3090	0.3024	0.2917	0.2827
19	0.3559	0.3189	0.2923	0.2860	0.2759	0.2668
20	0.3376	0.3025	0.2772	0.2713	0.2617	0.2525

TABLE 6.4.4. Comparison of exact and approximate critical points of V_1, V_2 and V_3 for α=0.05 and K = 3.

n	d	V_1		V_2		V_3	
		Exact	Approx.	Exact	Approx.	Exact	Approx.
10	6	.5012	.4853	.7417	.7740	.6953	.7258
10	12	.4426	.4156	.6262	.6371	.5899	.5991
10	18	.4249	.3968	.5920	.6004	.5588	.5652
10	24	.4163	.3881	.5756	.5833	.5439	.5494
20	6	.2435	.2358	.3602	.3758	.3376	.3524
20	12	.2150	.2019	.3041	.3094	.2864	.2909
20	18	.2064	.1928	.2874	.2915	.2713	.2744
20	24	.2022	.1885	.2795	.2832	.2641	.2667
20	30	.1998	.1861	.2748	.2784	.2599	.2623
20	36	.1982	.1845	.2718	.2753	.2571	.2594
20	42	.1970	.1834	.2696	.2731	.2551	.2574
20	48	.1962	.1825	.2680	.2715	.2536	.2559
20	54	.1955	.1819	.2667	.2702	.2525	.2547

TABLE 6.5.1. Simulated power values of the tests V_1, V_2 and V_3 for $\alpha = 0.05$, $K = 3$ and $\theta_3 = 0$.

(n = 11)

θ_1	θ_2	d = 15			d = 27		
		V_1	V_2	V_3	V_1	V_2	V_3
.0	.0	.061	.052	.054	.061	.045	.045
.1	.0	.114	.058	.056	.113	.066	.065
.2	.0	.245	.101	.105	.248	.106	.101
.3	.0	.468	.233	.249	.504	.235	.257
.4	.0	.674	.361	.413	.730	.407	.450
.5	.0	.844	.558	.617	.897	.638	.691
.6	.0	.951	.739	.776	.966	.818	.864
.7	.0	.982	.864	.892	.990	.928	.941
.8	.0	.999	.949	.970	.998	.978	.986
.9	.0	.999	.978	.989	.999	.995	.996
.2	.2	.147	.108	.076	.149	.119	.083
.4	.2	.537	.285	.309	.592	.326	.342
.6	.2	.880	.608	.652	.913	.679	.734
.8	.2	.992	.883	.920	.993	.951	.964
.4	.4	.529	.369	.304	.557	.409	.345
.6	.4	.844	.611	.656	.868	.677	.711
.8	.4	.972	.871	.893	.980	.911	.935
.6	.6	.850	.710	.655	.871	.770	.716
.8	.6	.960	.881	.897	.968	.926	.934
.8	.8	.965	.913	.904	.978	.940	.925

TABLE 6.5.1. Contd.

(n = 16)

θ_1	θ_2	d = 18 V_1	d = 18 V_2	d = 18 V_3	d = 42 V_1	d = 42 V_2	d = 42 V_3
.0	.0	.054	.046	.050	.058	.050	.056
.1	.0	.166	.079	.086	.170	.080	.085
.2	.0	.442	.209	.239	.482	.238	.264
.3	.0	.787	.468	.519	.838	.540	.591
.4	.0	.949	.751	.795	.973	.837	.873
.5	.0	.994	.928	.949	.997	.975	.987
.6	.0	.999	.988	.992	.999	.996	.998
.2	.2	.322	.215	.170	.336	.244	.186
.4	.2	.868	.638	.669	.891	.717	.754
.6	.2	.996	.964	.977	.998	.984	.990
.4	.4	.842	.729	.669	.876	.787	.746
.6	.4	.992	.944	.962	.996	.974	.979
.6	.6	.983	.957	.947	.992	.980	.972
.8	.6	.998	.987	.991	.999	.995	.996
.8	.8	.999	.996	.999	.999	.999	.999

TABLE 6.5.2. Power values of the statistics T_1 and V_1 for $\alpha = 0.05$, $\theta_3 = 0$, $K = 3$ and $n_1 = n_2 = n_3 = n = 16$.

θ_1	θ_2	d = 6		d = 18		d = 30		d = 42	
		T_1	V_1^*	T_1	V_1^*	T_1	V_1^*	T_1	V_1^*
.0	.0	.050	.046	.050	.054	.050	.062	.050	.068
.1	.0	.236	.146	.247	.166	.248	.169	.248	.170
.2	.0	.625	.375	.779	.442	.824	.487	.845	.489
.3	.0	.890	.620	.984	.787	.991	.814	.993	.838
.4	.0	.978	.821	.999	.949	.999	.978	.999	.978
.2	.2	.516	.281	.633	.322	.667	.333	.634	.336
.4	.2	.948	.745	.989	.868	.992	.887	.994	.891
.4	.4	.943	.734	.983	.842	.986	.872	.987	.876

*Simulated powers based on 1000 samples for each sample size.

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APPENDIX A

SOLUTION OF THE ML EQUATIONS UNDER THE HYPOTHESIS $\theta_1=\theta_2=\theta$

For a given set of ordered sample observations $x_{r_1+1} \leq x_{r_1+2} \leq \cdots \leq x_{n_1-s_1}$ and $y_{r_2+1} \leq y_{r_2+2} \leq \cdots \leq y_{n_2-s_2}$, an algorithm is presented to compute the ML estimates of location parameter θ and scale parameter σ under the hypothesis $\theta_1=\theta_2=\theta$. In Section 2.6, for $x_{r_1+1} \leq y_{r_2+1}$ and for different values of r_1 and r_2, the ML estimates $\hat{\theta}$ and $\hat{\sigma}$ of θ and σ are given as follows :

(a) <u>Case : $r_1=0, r_2=0$</u>

$$\hat{\theta} = x_{r_1+1} \text{ and } \hat{\sigma} = P/d^*,$$

(b) <u>Case : $r_1 > 0$, $r_2=0$</u>

$$\hat{\theta} = x_{r_1+1}-\hat{\sigma} \log (1+r_1/f) \text{ and } \hat{\sigma} = P/d^*,$$

(c) <u>Case : $r_1=0, r_2 > 0$</u>

(i) if $a \leq b$, then $\hat{\theta} = y_{r_2+1}-\hat{\sigma} \log (1+r_2/f)$ and $\hat{\sigma} = (P-fQ)/d^*$

(ii) if $a > b$, then $\hat{\theta} = x_{r_1+1}$ and $\hat{\sigma}$ is the solution of equation

$$\exp (Q/\hat{\sigma}) = 1 + r_2Q/(P-d^*\hat{\sigma}), \tag{A.1}$$

(d) <u>Case : $r_1 > 0$, $r_2 > 0$</u>

$$\hat{\theta} = x_{r_1+1}-\hat{\sigma} \log \{1+r_1Q/(d^*\hat{\sigma}+fQ-P)\}$$

and $\hat{\sigma}$ is the solution of the equation

$$\text{(A.2)} \qquad \exp(Q/\hat{\sigma}) = \{1 + \frac{r_2 Q}{P-d^*\hat{\sigma}}\}/\{1 + \frac{r_1 Q}{d^*\hat{\sigma}+fQ-P}\},$$

where $P = \sum_{i=r_1+1}^{n_1-s_1} x_i + s_1 x_{n_1-s_1} + \sum_{j=r_2+1}^{n_2-s_2} y_j + s_2 y_{n_2-s_2} - f\, x_{r_1+1},$

$Q = y_{r_2+1} - x_{r_1+1}$, $f = n_1+n_2-r_1-r_2$, $d^* = f-s_1-s_2$,

$a = Q/\log(1+r_2/f)$ and $b = (P-fQ)/d^*$.

As remarked in Section 2.6, equations (A.1) and (A.2) are solved by Newton-Raphson method with $(2P-fQ)/2d^*$ as an initial value.

The subroutine ESTMAT computes the ML estimates THETA and SIGMA of θ and σ respectively for given values of n_1, n_2, r_1, r_2, s_1 and s_2, and vectors X and Y. For this subprogram, the required accuracy AC, and the expected number of iterations NI are supplied from the main calling program. The failure indicator FI takes the value 1, if the iteration procedure does not converge in NI steps.

LANGUAGE

Fortran 10

STRUCTURE

SUBROUTINE ESTMAT(N1,N2,R1,R2,S1,S2,X,Y,NI;AC,THETA,SIGMA;FI)

Formal parameters

N1	Integer	input	: size of the first sample
N2	Integer	input	: size of the second sample
R1	Integer	input	: number of smallest observations missing in the first sample
R2	Integer	input	: number of smallest observations missing in the second sample
S1	Integer	input	: number of largest observations missing in the first sample
S2	Integer	input	: number of largest observations missing in the second sample
X	Real	input vector of length N1-R1-S1	: available ordered observations in the first sample
Y	Real	input vector of length N2-R2-S2	: available ordered observations in the second sample
NI	Integer	input	: upper bound for the number of iterations in which the iteration process is expected to converge
AC	Real	input	: desired accuracy
THETA	Real	output	: ML estimate of θ
SIGMA	Real	output	: ML estimate of σ
FI	Integer	output	: failure indicator

$$= \begin{cases} 1 & \text{if the iteration does not converge} \\ 0 & \text{otherwise.} \end{cases}$$

```
C        THIS SUBROUTINE COMPUTES THE ML ESTIMATES OF THETA
C        AND SIGMA BASED ON TYPE II DOUBLY CENSORED
C        SAMPLES FROM TWO EXPONENTIAL DISTRIBUTIONS
C
         SUBROUTINE ESTMAT (N1,N2,R1,R2,S1,S2,X,Y,NI,AC,THETA,
         1 SIGMA,FI)
         INTEGER R1,R2,S1,S2,FI
         DOUBLE PRECISION F,D,P,Q,A,B,X(999),Y(999),Z(999),S(999),
         1THETA,SIGMA,SUM,AC,U1,U2,U3,U4,FN,FP
         IF(X(R1+1).LE.Y(R2+1)) GO TO 40
C        INTERCHANGING THE SAMPLES
C
         N11=N1;NR1=R1;NS1=S1
         N1=N2;R1=R2;S1=S2
         N2=N11;R2=NR1;S2=NS1
         DO 10 I=R1+1,N1-S1
  10     Z(I)=Y(I)
         DO 20 I=R2+1,N2-S2
  20     Y(I)=X(I)
         DO 30 I=R1+1,N1-S1
  30     X(I)=Z(I)
C        CALCULATION OF F,D AND Q
C
  40     F=N1+N2-R1-R2
         D=F-S1-S2
         Q=Y(R2+1)-X(R1+1)
```

```
C       SUM OF ALL OBSERVATIONS
C
        SUM=O
        DO 50 I=R1+1,N1-S1
 50     SUM=SUM+X(I)
        DO 60 I=R2+1,N2-S2
 60     SUM=SUM+Y(I)
C       CALCULATION OF P
C
        P=SUM+S1*X(N1-S1)+S2*Y(N2-S2)-F*X(R1+1)
C       CALCULATION OF A AND B
C
        A=Q/DLOG(1.+R2/F)
        B=(P-F*Q)/D
C       SEPARATION OF DIFFERENT CASES
C
        IF(R2.EQ.O) GO TO 75
        IF((R1.EQ.O).AND.(A.LE.B)) GO TO 80
C       ESTIMATING SIGMA BY NEWTON-RAPHSON METHOD
C       WITH INITIAL VALUE S(1)
C
        S(1)=(2.*P-F*Q)/(2.*D)
        I=1
 65     U1=1.+R2*Q/(P-D*S(I))
        U2=1.+R1*Q/(D*S(I)+F*Q-P)
        U3=R2*Q*D/((P-D*S(I))**2.)
        U4=R1*Q*D/((D*S(I)+F*Q-P)**2.)
```

```
C       THE FUNCTION OF SIGMA IS DENOTED BY "FN"
C       AND ITS DERIVATIVE IS BY "FP"
C
        FN=DEXP(Q/S(I))-U1/U2
        FP=-DEXP(Q/S(I))*Q/(S(I)**2.)-(U3*U2+U1*U4)/(U2 **2.)
        S(I+1)=S(I)-FN/FP
        IF(DABS(S(I+1)-S(I)).LE.AC) GO TO 70
        IF(I.GE.NI) GO TO 85
        I=I+1
        GO TO 65
C       FINAL VALUE OF SIGMA AND THETA
C
 70     SIGMA=S(I+1)
        THETA=X(R1+1)-SIGMA*DLOG(1.+R1*Q/(D*SIGMA+F*Q-P))
        GO TO 90
 75     SIGMA=P/D

        THETA=X(R1+1)-SIGMA*DLOG(1.+R1/F)

        GO TO 90
 80     SIGMA=(P-F*Q)/D
        THETA=Y(R2+1)-SIGMA*DLOG(1.+R2/F)
        GO TO 90
C       ASSIGNING THE VALUE TO THE FAILURE INDICATOR
C
 85     FI=1
 90     RETURN
        END
```

```
C
C                      * * * * * * * *
                       CALLING PROGRAM
C                      * * * * * * * *
C
C     SOLUTION OF THE ML EQUATIONS
      INTEGER R1,R2,S1,S2,FI
      DOUBLE PRECISION X(100),Y(100),AC,THETA,SIGMA
C     INPUT VALUES,THE DATA X(I) AND Y(J) ARE SIMULATED
C     VALUES FROM E(2,1) DISTRIBUTION ARRANGED IN AN ASCENDING
C     ORDER OF MAGNITUDE
C
      DATA N1,N2,R1,R2,S1,S2,NI,AC,(X(I),I=4,14),(Y(J),J=3,12)/
     1 17,13,3,2,3,1,20,.00001,2.16140,2.21918,2.30073,2.84808,
     1 2.91879,3.14132,3.21995,3.34728,3.35224,3.45513,3.62251,
     1 2.15214,2.18279,2.29610,2.30496,2.41608,2.53415,2.57010,
     1 2.95275,2.96800,4.32659/
      CALL ESTMAT(N1,N2,R1,R2,S1,S2,X,Y,NI,AC,THETA,SIGMA,FI)
      IF(FI-1) 10,30,10
   10 PRINT 20,THETA,SIGMA
   20 FORMAT(10X,'THE ESTIMATE OF THETA IS',F9.5/10X,'THE
     1 ESTIMATE OF SIGMA IS',F9.5)
      GO TO 50
   30 PRINT 40,NI
   40 FORMAT(5X,'THE ITERATION DOES NOT CONVERGE IN',I4,'STEPS')
   50 STOP
      END
```

OUTPUT

```
THE ESTIMATE OF THETA IS 1.97921
THE ESTIMATE OF SIGMA IS 0.97834
```

APPENDIX B

EVALUATION OF $Q_d(x|s)$ AND $L_d(x|s)$

The procedures for evaluating the functions $Q_d(x|s)$ and $L_d(x|s)$, where d is a positive integer are presented. Function subprograms QD(D,X,S) and LD(D,X,S) are given for calculating these functions.

From Section 3.1, we have

$$Q_d(x|s) = \int_x^\infty e^{-y(1+s)} y^{d-1} \, dy/(d-1)! \quad , \; x \geq 0.$$

Note that, this integral is convergent only for $(1+s) > 0$, that is for $s > -1$. By substituting $z = y(1+s)$, we get

$$Q_d(x|s) = \int_{x(1+s)}^\infty e^{-z} z^{d-1} \, dz/\{(1+s)^d(d-1)!\}.$$

$$(B.1) \qquad = \sum_{j=0}^{d-1} e^{-x(1+s)} x^j (1+s)^{j-d}/j! \, , \; s > -1.$$

Similarly, for finite positive values of x, we have

$$L_d(x|s) = \int_0^x e^{-y(1-s)} y^{d-1} \, dy/(d-1)!.$$

This converges for all s, but has to be treated **separately** for $s = 1$ and $s \neq 1$ cases. Thus

$$(B.2) \qquad L_d(x|s) = \begin{cases} x^d/d! \, , & s = 1 \\ \left[1 - \sum_{j=0}^{d-1} e^{-x(1-s)} \{x(1-s)\}^j/j!\right]/(1-s)^d, & s \neq 1. \end{cases}$$

The FUNCTION QD(D,X,S) computes the summation given in equation (B.1) for D = 1, 2, ...; X $\geq$ 0 and S > -1. For D = 1, 2, ...; X $\geq$ 0 and $-\infty$ < S < ∞, FUNCTION LD(D,X,S) computes the value of $L_d(x|s)$ given in equation (B.2).

LANGUAGE

Fortran 10

```
C    FUNCTION SUBPROGRAM FOR COMPUTING QD(X|S)
C
      DOUBLE PRECISION FUNCTION QD(D,X,S)
      DOUBLE PRECISION SUM,X,S,X1,FAC
      INTEGER D
      SUM=0.0
      DO 10 J=0,D-1
      X1=1.0;I=J
      IF(J.EQ.0) GO TO 90
      X1=X*(1.+S)
      IF(J.EQ.1) GO TO 90
      DO 80 K=2,I
      FAC=DFLOAT(K)
 80   X1=X1*X*(1.+S)/FAC
 90   SUM=SUM+X1
 10   CONTINUE
      QD=DEXP(-X*(1.+S))*SUM/((1.+S)**D)
      RETURN
      END
```

```
C     FUNCTION SUBPROGRAM FOR COMPUTING LD(X|S)
C
      DOUBLE PRECISION FUNCTION LD(D,X,S)
      DOUBLE PRECISION X,S,SUM,X1,FAC,S1
      INTEGER D
      IF(S.EQ.1) GO TO 21
      SUM=0.0
      DO 15 J=0,D-1
      X1=1.0;I=J
      IF(J.EQ.0) GO TO 90
      X1=X*(1.-S)
      IF(J.EQ.1) GO TO 90
      DO 80 K=2,I
      FAC=DFLOAT(K)
   80 X1=X1*X*(1.-S)/FAC
   90 SUM=SUM+X1
   15 CONTINUE
      LD=(1.-DEXP(-X*(1.-S))*SUM)/((1.-S)**D)
      GO TO 23
   21 S1=1.
      DO 25 J=1,D
      FAC=DFLOAT(J)
   25 S1=S1*X/FAC
      LD=S1
   23 RETURN
      END
```